

# spooky halloween contest

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## **1 Introduction**

## Individual round

- How many ways can you arrange the letters in the word AKINFENWA such that no two vowels are next to each other?
- There is a unique monic polynomial  $p(x)$  satisfying  $(x - 81)p(3x) = 81(x - 1)p(x)$ . Find the sum of the roots of  $p(x)$ .
- In triangle  $ABC$ ,  $AB = 5$ ,  $BC = 7$ ,  $AC = 8$ , points  $D$  and  $E$  are chosen such that  $ABD$  and  $ACE$  are equilateral, also  $AD > CD$ , and  $AE > BE$ . What is the length  $DE$ ?
- Find the average value of  $|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|$  for all permutations  $a_1, a_2, \dots, a_{10}$  of the integers  $1, 2, \dots, 10$ .
- How many solutions are there to  $A+BC+DEF=GHIJ$  where placing next to each other means concatenation, and every digit is different?
- Consider polynomial  $P(x) = (1 + x + x^2 + \dots + x^{2018})^2 - x^{2018}$ . Each root can be written in the form  $z_k = r_k e^{i\theta_k}$  where  $0 \leq \theta_k < 2\pi$  and  $r_k \geq 0$ . Find the sum of the two least positive values of  $\theta$  in radians.
- Consider a black and white coloring of a 20 by 18 grid. Let  $n$  be the number of colorings such that every row and column has an even number of black squares. Find  $\log_2 n$ .
- Let  $n$  be the greatest possible value of  $abc$  given that  $2^{98} \cdot 63a + 2^{99} \cdot 94b + 2^{100} \cdot 63 \cdot 94c$  is a multiple of 1000, and  $a + b + c < 249$ . What is the value of  $n - (\lfloor \sqrt[3]{n} \rfloor)^3$ ?
- Let  $x, y, z$  be nonnegative real numbers such that  $x + y + z = 1$ . Find the minimum value of  $xy + yz + zx - 2xyz$ .
- Consider polynomial  $P(x) = x + 2x^2 + 3x^3 + \dots + 2017x^{2017} + 2018x^{2018} + 2017x^{2019} + 2016x^{2020} + \dots + x^{4035}$ . Let  $z_1, z_2, \dots, z_r$  be the distinct zeroes of  $P(x)$ . For  $k = 1, 2, 3, \dots, r$ ,  $z_k^2$  can be written as  $z_k^2 = r_k e^{i\theta_k}$ , where  $0 < \theta_k \leq 2\pi$  and  $\theta_k \leq \theta_{k+1}$  for any  $k < r$ . Find  $\theta_1 + \theta_2 + \theta_3$  in radians.
- The Triplets are playing a basketball game against the Twins in the 57th diep.io Basketball Championships. During each period the Triplets take three 2-point shots, each of which they have a  $\frac{1}{2}$  chance of making, and the Twins take two 3-point shots each of which they have a  $\frac{1}{3}$  chance of making. The game lasts 1 period, if the game is tied after a period both teams play another period where they follow the same pattern. What is the probability the Twins beat the Triplets?
- Let  $a$  and  $b$  be real numbers such that the complex solutions of  $x^3 + ax^2 + bx - 1$  all have magnitude greater than or equal to 1. Find  $a + b$  given that not all three roots are real.
- Consider regular pentagon  $A_1A_2A_3A_4A_5$  inscribed in a circle of radius  $\sqrt{\frac{3-\sqrt{5}}{4}}$ . Let point  $X$  be on the circle one half of the way between  $A_1$  and  $A_5$  on the minor arc of those two points. Find  $-XA_1 + XA_2 - XA_3 + XA_4$ .
- $\cos(\theta) \cos(3\theta) \cos(9\theta)$  can be written as  $\frac{\cos(a\theta) + \cos(b\theta) + \cos(c\theta) + \cos(d\theta)}{4}$ . Find  $a + b + c + d$ .

15. Vlad and Drac are playing a game of tennis. Vlad can return the ball with a probability of  $\frac{4}{7}$ , while Drac can return the ball with a probability of  $\frac{6}{7}$ . Drac serves the ball to Vlad. What is the expected number of returns that will occur in this round?

## Team round

Steve is entering a spooky dungeon with spooky challenges inside.

1. At the entrance, Steve sees a spiderweb in front of him whose strings are in the shape of a nonagon with all of its diagonals drawn in. To enter the dungeon, he makes 2 straight-line cuts in it with his sword, neither of them cutting through a vertex of the nonagon. What is the maximum number of strings that Steve could have cut through?(cutting the same string twice with his sword only counts as one string)
2. Steve finds 15 bags, each of which contains  $2^0, 2^1, 2^2, \dots, 2^{14}$  bones. He decides to split the 15 bags into two piles, with at least one bag in each pile. Let the total number of bones in the first pile be  $a$ , and the total number of bones in the second pile be  $b$ . What is the maximum value of the GCF of  $a$  and  $b$ ?
3. Steve sees a sinister shelf hanging in the hallway. It has the shape of a 5-6-7 triangle with its circumcircle and the sides of lengths 5 and 6 extended to meet the ground for support. Two circular mirrors are stuck between the boards of that formation, one below the circumcircle just barely touching the legs and the circumcircle, and another inside the circumcircle that also barely touches the legs and the circumcircle, in front of the side of length 7. Steve notices small notches on the rims of the mirrors exactly where the mirrors touch the circumcircle, so he takes a straight stick and finds that it fits perfectly in between the notches. Then, he sees the board of length 7 and the stick extend until they touch each other, and then from the top vertex of the triangle emerges a creepy beam of light that solidifies into a rod once it touches the intersection point of the board of length 7 and the stick. Find the length of this rod.
4. The rod then transforms into a snake with length 20 and negligible thickness. The snake has a status effect that allows it to only bend in 2 places in its body, and that makes the area in between any 2 points of its body glow. However, once the glowing area is maximized, the snake and the glowing area both transform into a portal. Find the area of the portal.
5. This portal leads to an infinitely long hall with ghosts. He sees a ghost with 6 HP and easily kills it, but he soon sees that each of the ghosts is more powerful than the previous ghosts, the  $n$ th ghost having  $HP_n = 50HP_{n-1} + HP_{n-2} + 24n + 15$ , if we assume there is a 0th ghost that has no HP. He has an enchanted sword with 37 damage that he can use to kill every ghost. However, on the 2019th ghost, the sword breaks, leaving that ghost alive with less than 37 HP. How much health does that ghost have remaining?
6. That ghost has created a challenge for Steve in which he has been given  $x$  indistinguishable arrows, and  $y$  distinguishable skeletons have spawned. The ghost now gives Steve a bow that always kills a skeleton in 3 hits. Steve has to shoot every arrow at the skeletons such that all skeletons remain alive, while hitting a skeleton with every arrow. Let  $S(x, y)$  be the number of ways he can do this. Find  $S(1, 2018) + S(4, 2018) + \dots + S(1000, 2018) + S(1003, 2018) + \dots + S(4036, 2018)$
7. Right after Steve completes the challenge, he finds himself stuck in the crossfire between two of the skeletons, one standing at  $A = (0, 10)$ , and the other standing at  $B = (6, 8)$ . The ghosts have put a fence blocking Steve from going past  $y = 0$ , but Steve has somehow found an axe that can break open the fence at one area, where he can then charge towards the

skeletons and kill them. To maximize his chances of surviving, he chooses the point where the skeletons' arrows hit him with the largest gap (in other words, maximizing  $\angle ACB$  where  $C$  is where Steve stands). Find the x-coordinate of where Steve breaks the fence.

8. Steve combines the bows of the skeletons into a cyclic 2018-gonal frame that has no 3 diagonals intersecting in a single point. Then he sees a spider crawl to a random vertex, and glide to another random vertex that it has not landed on, carrying a string of silk with it that fixes as a straight line between the 2 vertices. Then it repeats the process until it has been to every vertex once, in which it then glides back to the vertex it started at, completing a very weird-looking spider web that flashes rapidly for a while before stopping. What is the expected value of the number of intersections between the silk strings inside the frame?
9. In the distance, Steve sees a giant glass sphere that has been summoned by the spider finishing the web. In it are 3 spherical skulls of radius 6 meters, all externally tangent to each other, a giant spherical pumpkin of radius 7 meters, and internally tangent to the surface of the giant glass sphere. Find the radius of the giant glass sphere.
10. Steve realizes too late that the giant glass sphere is rolling towards him from the other end of the long hall that he is in one end of, and that he cannot escape from it. He runs with constant speed towards the other end of the hall, getting run over by the sphere when he is running. When either Steve or the sphere reaches an end of the hall, they turn around and run with the same constant speed the other way. The second time Steve gets run over by the sphere, he is 562 blocks away from where he started, and the third time he is finally killed by the sphere 278 blocks away from where he started running. Find the distance from Steve's starting point where the sphere hit him first.