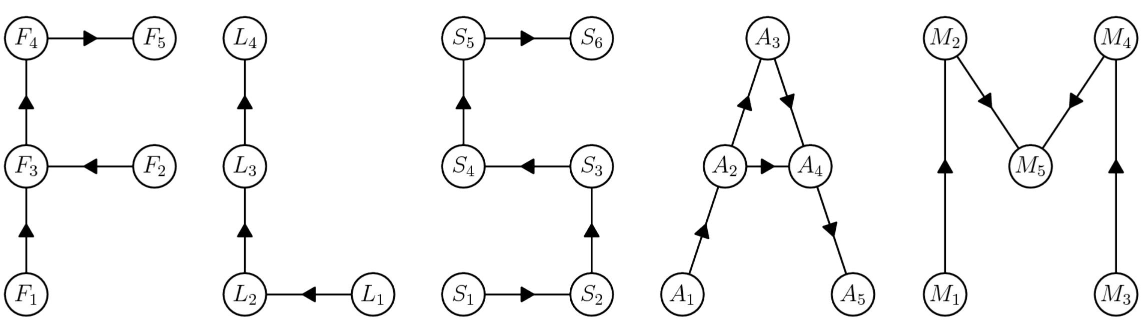
Florida Student Association of Mathematics

2019-2020 Introduction Meeting — August 31, 2019

Ultra Relay

Welcome to the **FLSAM Ultra Relay!** This event consists of 25 problems separated into five different relays. Many of the problems in these relays will depend on other problems, as indicated by the network below. Problems are not all worth the same number of points; the point value for each problem can be found next to the problem.

This round is run like the HMMT Guts Round or the PUMaC Live Round. When the time begins, each team will send a member to pick up copies of the first relay. Once a team has their answers, they send a student to turn in those answers and pick up the next relay. A team **may not** go back to a previous relay after turning in answers, so allocate your time effectively.



The label for each problem also refers to the answer of that problem. For example, F_1 denotes the answer to problem **F1**.

You will have ***45 minutes*** *to complete the test. Good luck, and have fun!*

F1. **[2]** Determine the value of $\frac{2019^2-2004^2}{2019+2004}$.

F2. **[2]** Bill is throwing balls at a row of 5 bottles. 3 of these bottles contain a prize, but Bill doesn't know which ones. Bill randomly selects a bottle and throws the ball at it. He has a $\frac{2}{3}$ chance of hitting the bottle he aims at and a $\frac{1}{3}$ change of hitting some other bottle (either way, he always hits one bottle). The probability that Bill wins a prize can be expressed in simplest form as $\frac{p}{q}$. Find $p+q$.

F3. **[4]** Let $ABCD$ be a rectangle such that $AB = F_1$ and $BC = F_2$. Suppose that M and N are the centers of the circles inscribed inside triangles $\triangle ABC$ and $\triangle ADC$ respectively. What is MN^2?

F4. **[5]** After an hour of attempting to decide what to eat, Joanne, Daniel, and 5 other students have decided to order Panda Express. Each person spends an integer number of dollars on their food, and in total they spend F_3 dollars. Let M be the highest amount of money any one of the 7 people pay. Find the smallest possible value of M, in dollars.

F5. **[7]** Let $\circledcirc$ be a binary operator such that $a \circledcirc b = ab - 3(a+b) + 12$. Evaluate

$$1 \circledcirc (2 \circledcirc (3 \circledcirc (\cdots ((F_4 - 1) \circledcirc F_4) \cdots))).$$

L1. **[3]** If p, q, and r are nonzero integers satisfying $p^2 + q^2 = r^2$, compute the smallest possible value of $(p+q+r)^2$.

L2. **[5]** Equilateral triangle ABC has side length L_1. A circle ω is inscribed inside $\triangle ABC$. A point P is selected randomly on the circumference of ω. The probability that $\angle BPC$ is acute can be expressed in simplest form as $\frac{p}{q}$. Find $p+q$.

L3. **[7]** A machine inputs a word and outputs the result of swapping any two of its letters. For example, it could turn CARP into CRAP. Given the $2L_2+1$-letter word TNTNTNTNT...NT with L_2+1 T's and L_2 N's, how many different words could be the result of using the machine twice on this word?

L4. **[10]** Two circles, ω_1 and ω_2, have radii 1 and L_3 respectively, and are externally tangent at point P. A common external tangent of ω_1 and ω_2 meets ω_1 at Q and ω_2 at R. Find the area of $\triangle PQR$.

S1. **[2]** If a thirty-second note is a thirty second note, how many minutes long is a quarter note?

S2. **[3]** It's finally Taco Tuesday and Kev wants to celebrate by serving tacos! He has two types of shell, three types of meat, and S_1 types of vegetables available. Carol and Michelle each make a taco by randomly selecting one shell, one meat, and one vegetable. The probability that their tacos are different can be expressed in simplest form as $\frac{p}{q}$. What is $p+q$?

S3. **[4]** Let T be the absolute value of the difference of the tens and units digits of S_2. Find the smallest positive integer n such that $n^2 + 7n + 6$ is divisible by 5^T.

S4. **[6]** Let T be the product of the digits of S_3. A regular tetrahedron has volume n. The centers of the four faces are connected to form another tetrahedron. If the volume of this new tetrahedron can be expressed in simplest form as $\frac{p}{q}$, what is $p+q$?

S5. **[7]** A circle Ω centered at the origin has radius $4 \cdot S_4$ and point P is located at $(8,4)$. If the area enclosed inside the locus of all midpoints of chords of Ω that pass through P can be expressed as $k\pi$ for some positive integer k, find k.

S6. **[8]** Dhruva and Michael planned to meet at a coffee shop. However, neither of them can remember what time they agreed on! Each of them will arrive at a random time between 1 pm and 2 pm. Dhruva will stay for 30 minutes, while Michael will only stay for S_5 minutes (he needs to get back to his anime Minecraft server). What is the probability that they will both be in the coffee shop at the same time?

A1. **[4]** Two primes p_1 and p_2 satisfy $p_1 + p_2 = 15$. What is the sum of all possible values of $p_1 p_2$?

A2. **[6]** A bin contains no more than A_1 balls, all of which are either red or blue. If one red ball and one blue ball are removed from the bin, the probability that a randomly selected ball is red becomes $\frac{3}{5}$. Let m be the maximum number of balls there could have initially been. Find the sum of the digits of m.

A3. **[7]** Two concentric squares, one of side length 3 and the other of side length A_2, are placed so that corresponding edges are parallel. A third square is placed so that it is inscribed inside the outer square but circumscribed around the inner square. What is the area of this third square?

A4. **[8]** Connor and Arnav play a game involving the function $f(x) = A_3 x^3 + ax^2 + bx + A_2$. Arnav states a rational number r, and Connor has to find integers a and b such that $f(r) = -1$. How many choices does Arnav have for r such that Connor will be able to find a working pair (a, b)?

A5. **[10]** Let $T = A_4$. Mr. Duck opens a fortune cookie and finds that the lucky numbers are 3, 5, 18, T, 37, and 45. Find the sum of all primes p for which the lucky expression

$$3^{p^3} + 5^{p^5} + 18^{p^{18}} + T^{p^T} + 37^{p^{37}} + 45^{p^{45}}$$

is divisible by p.

M1. **[6]** Steve's inventory is filled with random tools and weapons. He has Γ diamond hoes, Σ bows, Υ diamond pickaxes, and Ψ tridents. Steve used the same amount of diamonds crafting his hoes as his pickaxes, and he used as many sticks crafting the bows as his hoes and pickaxes combined.[1] Each of the 37 slots of his inventory are filled with exactly one of these items. Assuming Γ, Σ, Υ, and Ψ are all positive integers, find $\Sigma \cdot (\Psi - \Upsilon) + \Gamma$.

M2. **[6]** To craft the bows, hoes, and pickaxes in **M1**, Steve used $\diamondsuit$ diamonds, $\setminus$ sticks, and $\wr$ string. Find the value of $\frac{\setminus}{\diamondsuit - \wr}$.

M3. **[8]** A conduit in Minecraft placed at (64,64,64) is powered by prismarine blocks around it that are arranged in a certain configuration. To power the conduit, all of the prismarine blocks must:

- Have 1 coordinate be 64
- Have another coordinate be 2 from 64
- Have the third coordinate be at most 2 from 64
- Have all 3 coordinates be integers

When the maximum number of prismarine blocks are placed, the conduit is fully powered. How many prismarine blocks does this require?

M4. **[8]** Define the *surrounding blocks* of a lattice point P to be the 26 lattice points whose space distance from P is less than 2. A surrounding block of the conduit is randomly chosen and labeled W. What is the expected value of the number of prismarine blocks that are surrounding blocks of W, assuming the fully powered conduit from **M3**?

M5. **[12]** Let $T = M_2 \cdot M_4$. Let a *rail* be a 1x1 square that can be placed on the Cartesian plane. Steve starts with T rails in his hand. He places 1 rail centered at $(0, 0)$ with its sides parallel to the x and y axes. Then he places the rest of the rails one at a time, with the restriction that each rail must completely share an edge with a previously placed rail. What is the farthest distance a rail can be centered from the origin?

[1] Diamond hoes are crafted with 2 diamonds and 2 sticks, diamond pickaxes with 3 diamonds and 2 sticks, and bows with 3 sticks and 3 string.