Florida Student Association of Mathematics

2020-2021 Florida Online Math Open

December 26th, 2020 - January 1st, 2021

Problem Contributors

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Thank you to everyone who submitted problems!

Rules

1. This is a 50 problem, team-based competition that takes place over four days. Answers must be submitted before 7PM EDT on January 1st.

2. This is a team event; collaboration with your teammates is encouraged. Discussion about the contest or its problems is not allowed between members of different teams or anyone not competing.

3. Each correct answer is worth 1 point. Incorrect or omitted answers are both worth 0 points.

4. The difficulty of each question will be determined by how many teams correctly answer it. Questions with fewer solves will be considered more difficult. Ties will be broken by considering the most difficult question each team correctly answers. The team that correctly answered the more difficult question wins the tiebreaker. If necessary, the process will move to the second most difficult question each team correctly answers and so on.

5. You may use four function calculators. All problems on this test can be solved without a calculator. Drawing aids such as graph paper, rulers, compasses, and protractors are permitted, but electronic programs such as graphing calculators, Desmos, or Geogebra are not.

6. **All answers are non-negative integers**. Any non-integer quantities will be converted to integers by instructions in the problem. Fractions of the form $\frac{m}{n}$ and radicals of the form $p\sqrt{q}$ are assumed to be in simplest form (That is, $\gcd(m, n) = 1$, q is squarefree, and all variables are positive integers). Isolated radicals of the form $\sqrt{k}$ do not necessarily follow this rule.

7. Make sure to check both the `#flomo-updates` channel in Discord and emails from `flsamflorida@gmail.com` for any clarifications or updates. If you have any questions, please contact the testing coordinators at `flsamflorida@gmail.com` or `albert.wang.2004@gmail.com` or through Discord DMs to the given tags.

In addition, the following approximations and identities may be useful:

$$\pi \approx 3.14159265359$$
$$e \approx 2.71828182846$$

Problems

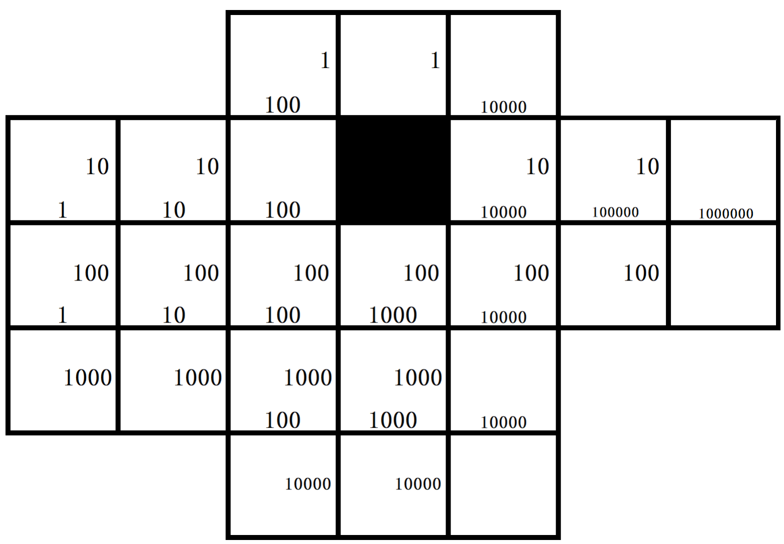
1. Integers F, L, S, A, M form a strictly increasing arithmetic sequence in that order, and $F^2 = A^2 = 36$. What is M?

2. Square $ABCD$ with side length 100 has midpoint M of $\overline{BC}$, and N is the midpoint of $\overline{AM}$. What is the area of $\triangle ADN$?

3. While Charley isn't looking, Aaron is attempting to use two of Charley's Charmanders to cook a steak. The first Charmander is able to finish by itself in 30 minutes, and the second Charmander is able to finish by itself in 20 minutes. Initially, Aaron has both charmanders cooking, but five minutes in, the first Charmander runs away. Assuming that both Charmanders work at a constant rate, the length of time that Aaron must wait for the second Charmander to finish the task alone can be expressed as $\frac{m}{n}$ minutes, with the fraction $\frac{m}{n}$ simplified and m, n positive integers. Compute $100m + n$.

4. How many integers n are there such that n^2 is a three-digit integer?

5. On the grid below, partition the white squares by cutting along the black gridlines to produce three congruent polygons, and minimize the sum of the weights of the gridlines that are cut. Find this minimum value. There is exactly one way to partition the figure.



6. How many contiguous subsequences of "BANANANANA" contain at least two A's? Sequences that are translations of each other are considered different; for instance, "BAN$\boxed{\text{ANA}}$NANA" and "BANAN$\boxed{\text{ANA}}$NA" are considered two distinct subsequences.

7. The number of candy canes C in a house depends on the number of ornaments R in the same house such that
$$\log_{10}(\log_{10} R) = 5 \cdot C^{\log_{10} C}.$$
If $R = 10^{10^{50}}$ in Santa's house, find the sum of the possible values of $10C$. Note: C can be any real number.

8. Two distinct squares each with area 100 are positioned such that two adjacent vertices of the second square lie on two adjacent edges of the first square; furthermore, three edges of the second square lie outside the first square. Find the minimum area that both squares can cover.

9. Find the sum of all positive integers $k \leq 10$ that divide
$$111,111,111,111,111,111,111_{64}.$$

10. Saathvik has discovered a new material which can change freely from solid to liquid. He half-fills a cylinder with radius 5 and height 10 using this material in liquid form, and tilts it such that the surface of the material is an ellipse and has maximum area. From here, it is cast to a solid. What is the square of the height of this new solid when taken from the elliptical base?

11. Aaron and Alex have competing units of measure, named the *Aaron (aa)* and the *Alex (ax)*, respectively. Both have produced rulers; Aaron has a 30aa ruler, and Alex has a 30ax ruler, both evenly numbered 0 to 30 with marks one unit apart. They tape the rulers such that Aaron's 6 and 9 marks match with Alex's 25 and 21 marks, respectively. If the combined ruler now has length 1 meter, then there exists a simplified fraction $\frac{m}{n}$ for which

$$\frac{m}{n}\text{meters} = 1\text{aa} + 1\text{ax}.$$

Find $100m + n$.

12. Neglecting the social distancing measures in a mall, Jae has accidentally found himself at $(0,0)$ in a square room with vertices $(0,0),(10,0),(10,10),(0,10)$. Positioned at $(0,10),(5,5),(10,5)$ are three infected shop-goers. From $(0,0)$, Jae is attempting to reach the exit at $(10,10)$, and can't walk through any other wall in the room. Jae's chance of infection is the reciprocal of his minimal distance to one of these infected locations. The minimal distance Jae must travel to obtain the minimal infection chance is some real number d. Find $\lfloor 10^4 d \rfloor$.

13. Charley has 1000 cubes numbered $1,\ldots,1000$ and stacks them to form a $10\times 10\times 10$ cube. For every pair of cubes that share a face, he writes down their sum, and then he adds up all these values. What is the minimum final number he can get?

14. In $\triangle ABC$, point D is on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC = 150^\circ$. Also, $AC = 2$, $AB = 5$. Let the ratio of the area of $\triangle ABD$ to the area of $\triangle ACD$ be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

15. Given $x^2 + y^2 = 1$ for positive reals x, y and $x \neq y^2$, let the minimum value of $\frac{x^2+x-y^2-y^4}{x^2-y^4}$ be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

16. Iris and Divij are walking on the six lattice points $(0,0),(1,0),(2,0),(0,1),(1,1),(2,1)$. Iris starts at $(0,0)$, Divij starts at $(1,1)$ and every second they randomly walk 1 unit away to another one of these lattice points until they meet. The probability that they meet on a lattice point (x,y) with $x+y$ odd is a simplified fraction $\frac{m}{n}$. Find $100m + n$.

17. Consider the minimal positive real solution $0 < x < 0.2$ to the equation

$$x(1-x(1-x(1-x(1-x(1-x(1-x(1-x))))))) = \frac{1}{10}.$$

Find $\lfloor 10^4 x \rfloor$.

18. There are positive integers a, b, c, n with c not divisible by the square of any prime satisfying $x = -a + b\sqrt{c} + \log_2 n$ and

$$2^{2^x} = \sqrt{8}^{\sqrt{8}^{\sqrt{8}}}.$$

Find $10^3 a + 10^2 b + 10^1 c + 10^0 n$.

19. Consider the graph of $y = \sin x$ on a Cartesian plane with origin O. Let P be the intersection point with $y = x/37$ having the smallest positive x coordinate and Q be the intersection point with $y = x/29$ having the smallest positive x coordinate. The projection of P onto the x-axis is P'. Then, the ratio of the area of $\triangle OP'Q$ to the area of $\triangle OPQ$ is some simplified fraction $\frac{m}{n}$ for positive integers m, n. Find $100m + n$.

20. Saathvik starts at the point $(0, 2)$ on the coordinate plane and wants to visit $(2, 6)$ and $(5, 9)$ while picking up some goods from the river described by the line $x = y$. The square of the minimum distance that he could travel can be represented as a simplified expression $a + b\sqrt{c}$ for positive integers a, b, c with c squarefree. Find $10000a + 100b + c$.

21. A *yo' mama joke* is a sequence of 69 English letters starting with "YO". Furthermore, the number of times the substring "FAT" appears in the joke is defined as the *quality* of the joke. The expected quality of a random yo' mama joke can be written as a simplified fraction $\frac{m}{n}$ with m, n positive integers. Find $m + n$.

22. A rectangle $ABCD$ with $AB = CD = 8, BC = DA = 6$ has point E on AB such that $BE = BC = 6$. The radius of the largest circle that may be contained in quadrilateral $AECD$ can be written as a simplified expression $a - b\sqrt{c}$ for positive integers a, b, c. Find $100a + 10b + c$.

23. A *ring* is a circle ω such that some contiguous section of the circle's circumference is colored red and the rest colored green. An *ornament* is a collection of rings in the plane such that

- No three rings meet at one point,
- Every pair of intersecting rings intersects at two points,
- At any point where two rings meet, the colors are different.

Consider an ornament with 2020 rings. What's the maximum number of intersection in the ornament?

24. How many 15-digit numbers using only the digits $4, 5, 6$ have adjacent digits being relatively prime?

25. Albert and Vismay are playing a game using a six-sided die numbered from 1 to 6. First, Albert selects a binary string of length 2, and then Vismay selects a different binary string of length 2. Starting with a new empty sequence, if they roll a multiple of 3, they write down a 0, and otherwise they write down a 1. The first player to have their chosen sequence appear as a substring of the new sequence wins. If both players choose their sequence to maximize his probability of winning, the probability that Albert wins may be written as a simplified fraction $\frac{m}{n}$ for positive integers m, n. Compute $100m + n$.

26. Let the incircle of acute scalene triangle ABC be ω, with center I. Let $E = \omega \cap CA$ and $F = \omega \cap AB$. Then, reflect F over I to point Y. Line AY meets ω again at X. Given that $\angle FAX = 45^\circ, FY = 2$, and the area of triangle FEY can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

27. There are 5 points in a plane with no 3 of them being collinear. The lines connecting any two points are not parallel or perpendicular to each other. From each point, draw perpendicular lines to all the lines made by connecting the other 4 points in pairs. What is the maximum number of total intersection points between all pairs of perpendicular lines?

28. A circle intersects a parabola $y = x^2 + 1$ at 4 points in the xy-plane: $(-2, 5), (5, 26), (6, 37), (a, b)$. Find $a + b$.

29. Let a positive integer n be *strong* given that there does not exist positive integer $k < n$ such that $\tau(n) = \tau(k)$, where τ is the divisor function, outputting the number of divisors of its input. Find the smallest strong positive integer divisible by 7.

30. In an alternate reality, Divij is a farmer. Inconveniently, he must buy fences from Ritvik's Rigid Rails, which sells r, s-fences for any positive reals r, s, which are formed by $\overline{AB}, \overline{BC}$ such that $\angle ABC = 90^\circ$ and $AB = r, BC = s$. Using a $4, 4$-fence, a $5, 5$-fence, a $7, 7$-fence, and a $10, 10$-fence, Divij connects their endpoints into a non-self-intersecting polygon. What is the largest area that he can enclose?

31. Tom and Jerry are back to their antics. Jerry moves between three different holes in the wall, and Tom is trying to shine the flashlight on Jerry. At time $t = 1$, Jerry will be at any one of the holes with probability $1/3$. For integer $i > 1$, at time $t = i$, Jerry moves to a hole different from the one it was at for time $t = i - 1$, with probability $1/2$ of moving to either vacant hole. For each integer time $t = 1, 2, \ldots,$ Tom chooses one of the three holes and shines his flashlight into it. Let E be the expected time for Tom to shine the flashlight onto Jerry for the first time. Using the optimal strategy, the smallest possible value of E can be expressed as a simplified fraction m/n for positve integer m, n. Find $100m + n$.

32. Let a, b, c denote the roots of $x^3 + 69x^2 + 420x + 1337 = 0$. Given that $P(a) = \frac{b+c}{a}$, $P(b) = \frac{c+a}{b}$, $P(c) = \frac{a+b}{c}$, where P is a monic cubic polynomial, compute $\lfloor P(0) \rfloor$.

33. Two perpendicular segments $\overline{AC}, \overline{BD}$ lie in the plane with $\angle ABC + \angle ADC = 180^\circ$. If $AB = 25, BC = 39, CD = 60, DA = 52$, compute the area of concave $ABCD$.

34. Let $\{a_n\}$ be a sequence of positive integers such that $a_1 = 1$, and for all $n > 1$

$$a_n = \lfloor \sqrt{a_1 + a_2 + \cdots + a_{n-1}} \rfloor,$$

where $\lfloor x \rfloor$ is the largest positive integer less than equal to x. Compute a_{2020}.

35. Say a set of positive integers S is *amicable* if it is a non-empty subset of $\{1, 2, \ldots, 2020\}$ and the product of numbers of S is a perfect power of 10. What is the cardinality, or size, of the largest possible amicable set?

36. Consider the number $N = 273! + 273!!$. There exists a unique, three-digit prime p such that p^2 divides N. Find p.

(Note that $x!$ and $x!!$ denote the factorial and double factorial notations, where

$$x! = x(x-1)(x-2)\cdot \cdots \cdot 2 \cdot 1$$

$$x!! = x(x-2)(x-4)\cdot \cdots \cdot (a+2)\cdot a$$

where a is either 1 or 2, depending on whether x is odd or even, respectively.)

37. A *mountain* is a sequence $a_1, a_2, a_3, \ldots, a_{2020}$ of positive integers less than 2021 such that there exists a positive integer $1 \le k \le 2020$ where $\cdots < a_{k-2} < a_{k-1} < a_k = 2020 > a_{k+1} > a_{k+2} > \ldots$. There are M mountains. Find the largest power of 2 dividing M.

38. Let $a_1 = 1$, $a_2 = 46$, and the sequence $(a_n)_{n \ge 1}$ be described recursively by $a_n = 34a_{n-1} + 59a_{n-2}$. Given that $\sum_{k=1}^{\infty} \frac{a_k}{36^k} = \frac{m}{n}$ in simplest form, find $100m + n$.

39. Karthik writes the numbers $1, 2, \ldots, 2021$ are written on a board. A *move* consists of Karthik erasing two numbers a, b and replacing them with the value $\frac{ab}{a-b}$. After Karthik makes 2020 moves on the numbers on the board, the number M remains. The largest and second largest possible values of $\frac{1}{M}$ are M_1, M_2, respectively. Given that $M_1 - M_2$ can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers, find $100m + n$.

40. Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ which satisfies

$$f(a-1)^2 + f(b-1)^2 = 1 - 2f(a)f(b) + f(a+b)^2$$

for all integers a, b. Find the sum of all distinct possible values of $f(0) + f(1) + \cdots f(2020)$ that lie in the range $[-10^9, 10^9]$. If such f doesn't exist, answer 2020.

41. If a, b, c are real numbers such that $abc = a + b + c + 11$, find the minimum value of

$$(a^2+1)(b^2+1)(c^2+1).$$

42. Consider all triples of positive integers (x, y, z) that satisfy $x^2 + y^2 - z! = 2019$. Over all of these triples, find the maximum value of xyz.

43. Let $ABCDEF$ be a regular hexagon, where the points are labeled in that order. A certain Charmander starts at point A and moves randomly, going one vertex counterclockwise/clockwise with equal probability. Given that the expected number of times the Charmander reaches D before getting to B for the third time can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $100m+n$.

44. A *k-dropped base* $\mathfrak{B}$ is a series of positive integer weights $b_{k-1}, \ldots, b_2, b_1, b_0$ such that a series of positive integers $d_{k-1}, \ldots, d_2, d_1, d_0$ in base $\mathfrak{B}$ has

$$N = \overline{d_{k-1} \ldots d_2 d_1 d_0}_{\mathfrak{B}} = d_{k-1}b_{k-1} + \cdots + d_2 b_2 + d_1 b_1 + d_0 b_0.$$

A *fire* pair $(N, \mathfrak{B})$ with positive integer sequence N and k-dropped base $\mathfrak{B}$ is such that the sum of the squares of the digits of N equals the value of $N_{\mathfrak{B}}$. For all 3-dropped bases $\mathfrak{B} = b_2, b_1, b_0$ such that for both $N = 180, 30, 60$ and $N = 60, 180, 60$, $(N, \mathfrak{B})$ is *fire*, find the smallest positive value of $\overline{b_2 b_1 b_0}_{\mathfrak{B}}$.

45. Iris is attempting to fill in a 10×10 grid with natural numbers from 1 to 10 inclusive, as long as cells that share the same side or corner are coprime and no integer appears more than a times for some positive integer a. However, Ritvik is attempting to derail this by choosing a such that Iris cannot fill in her grid. Find the maximum value of a.

46. Find the largest positive integer $n > 1$ such that $n^2 + \phi(n)$ divides $n^3 + 1000$. (Note that $\phi(x)$ represents the number of positive integers less than x which are relatively prime to x.)

47. Over all ordered pairs of integers $(a, b) \neq (0, 0)$ that satisfy $a^3 + b^3 = 10^{\gcd(a,b)} - 1$, find the sum of all distinct possible values of $a^2 + b^2$ (For any two integers $(a, b) \neq (0, 0)$, $\gcd(a, b)$ denotes the largest positive integer which divides both a and b).

48. Triangle $\triangle ABC$ with $AB = 13, BC = 14, CA = 15$ has orthocenter H and circumcircle ω. The feet of the altitudes from A, B, C are D, E, F respectively. The nine point circle Γ, namely the circumcircle of $\triangle DEF$, has a tangent ℓ at D intersecting ω at P, Q. A non-degenerate circle passing through P, Q is tangent to Γ at point T. The value of AT^2 can be written as a simplified fraction $\frac{m}{n}$ for positive integers m, n. Find $100m + n$.

49. Last time, Iris overthrew Albert's ruthless dictatorship over a 10×10 grid. This time, Albert has returned on a 100×100 grid! However, Albert is extremely scared of *snakes*. Define a snake to be a sequence of cells $C_1, C_2, C_3, \ldots, C_{200}$ such that the following holds:

- Every row or column contains exactly two C_i.
- C_1, C_2 are in the same column, and likewise C_3, C_4 share a column, C_5, C_6 and so on up to C_{199}, C_{200}.
- C_2, C_3 are in the same row, and likewise C_4, C_5 share a row, $C_6 C_7$ and so on up to C_{200}, C_1.

Iris places one snake on the grid. A cell C in the grid is considered to be *scared* if any of the following hold:

- It is part of Iris's snake.
- There exists two cells C_i, C_j in the snake such that C_i, C, C_j appear in that order in the same row or column, forwards or backwards.

Find the maximum number of *scared* cells.

50. Consider $\triangle ABC$ with circumcenter O and orthocenter H. Let D denote the intersection of $\overline{AO}$ and $\overline{CH}$, and let $\odot(ODH)$ intersect $\overline{AH}$ a second time at E. Given that $AE = 41$, $AO = 19$, $AC = 28$, OH^2 can be written as a simplified fraction $\frac{m}{n}$ for positive integers m, n, compute $m + n$.