# Answer Key

**F1.** 42

**F2.** 4042

**F3.** 0

**F4.** 588

**F5.** 207

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**L1.** 841

**L2.** 119

**L3.** 148

**L4.** 295

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**S1.** $-2$

**S2.** 510

**S3.** 43

**S4.** 42

**S5.** 0

**S6.** 100

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**A1.** 39

**A2.** 14

**A3.** 39

**A4.** 777

**A5.** 301088

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**M1.** 49

**M2.** 65

**M3.** 108

**M4.** 179

**M5.** 35

**F1.** **[2]** Bill is 20 years older than Jill. If Jill's age divides Bill's age, what is the sum of all the possible values of Jill's age?

*Proposed by Sharvaa Selvan*

We have $n \mid n+20$, so $n \mid 20 = 2^2 \cdot 5$. The sum of all possible values of Jill's age is then $(1+2+4)(1+5) = \boxed{42}$.

**F2.** **[4]** For positive integers $a, b$, find the minimum possible integer value of $\frac{a^2+b^2}{2021}$.

*Proposed by Aaron Hu*

For $43 \mid a^2+b^2$ to hold true, we must have $43 \mid a, b$ from Fermat's Christmas Theorem. Similarly, we must have $47 \mid a, b$, so the answer is

$$\frac{2021^2+2021^2}{2021} = \boxed{4042},$$

as desired.

**F3.** **[5]** Sharvaa gets the lengths $F_1, F_2, F_2 - F_1$, and 1, and forms a cyclic quadrilateral in that order. Aaron gets the lengths $F_2, F_1, F_2 - F_1$, and 1, and forms a cyclic quadrilateral in that order. What is the maximum difference in the areas of the cyclic quadrilaterals they form?

*Proposed by Alex Li*

This is a troll that relies on the fact that all cyclic quadrilaterals with the same lengths (in any order!) have the same area. Therefore, the answer is $\boxed{0}$.

**F4.** **[6]** Alex, Aaron, and Charley play ping-pong, with a 2-person team vs. a 3-person team. Aaron and Charley form a team with probability $\frac{0}{10}$. If Aaron's team wins with probability $\frac{6}{7}$ per round and Charley always gets $\frac{23}{28}$ of the points for his team, find the expected number of rounds played for Charley to get to 69 points. Each round gives 1 point.

*Proposed by Alex Li*

Charley and Alex form a team against Aaron. Since Charley and Alex get $\frac{1}{7}$ of the total number of rounds and Charley receives $\frac{23}{28}$ of these points, Charley gets $\frac{23}{196}$ of the total number of points. Dividing 69 by this gives $\boxed{588}$ games.

**F5.** **[8]** Alex and Charley have a pile of $k$ rocks, where $k$ is a positive integer. They divide the pile of rocks into two smaller piles, and they each take one of the piles. They agree to each choose two rocks from their pile. Find the sum of all possible values of $k$ given that there are $\frac{F_4}{14}$ more ways that Alex can choose his two rocks than Charley can.

*Proposed by Aaron Hu*

Suppose that Alex's pile has $m$ rocks, and Charley's pile has $n$ rocks. Then we have $\binom{m}{2} - \binom{n}{2} = 42$, which is equivalent to

$$(m-n)(m+n-1) = 84.$$

We must have $m+n-1 > m-n$, so $m+n-1$ can equal any factor of 84 greater than $\sqrt{84}$. This gives the sum of all distinct values of $m+n = k$ as

$$(1+2+4)(1+3)(1+7) + 12 - (1+2+3+4+6+7) - 6 = \boxed{207},$$

as desired.

**L1.** [**3**] Let $ABCD$ be a square. Extend $\overline{AB}$ past $B$ to a point $A'$ Such that $AA' = 21 \cdot AB$. Similarly, extend $\overline{BC}$ past $C$, $\overline{CD}$ past $D$, and $\overline{DA}$ past $A$ to points $B'$, $C'$, and $D'$ respectively such that $BB' = 21 \cdot BC$, $CC' = 21 \cdot CD$, and $DD' = 21 \cdot DA$. Compute the ratio of the area of $A'B'C'D'$ to the area of $ABCD$.

*Proposed by Aaron Hu*

WLOG assume that $ABCD$ is a unit square, and consider right triangle $\triangle A'BB'$. Then

$$A'B = AA' - AB = 20 \cdot AB = 20, \quad BB' = 21 \cdot BC = 21,$$

so $A'B' = 29$. From symmetry, $A'B'C'D'$ is a square, so the answer is $29^2 = \boxed{841}$.

**L2.** [**6**] Let $T$ be the lefmost digit of $L_1$. Palm Harbor University High School wants to change their name to be a string with $T$ characters. Each character will be either $a, b$, or $c$ and the following conditions are met:

- The first character is an $a$.
- The last character is a $c$.
- An $a$ and a $c$ never appear consecutively.

Find the number of possible strings.

*Proposed by Aaron Hu*

Let $a_n$ denote the number of sequences length $n$ starting with 1, ending in 1, with no $1'$s and $3'$s are adjacent. Similarly define $b_n, c_n$ for ending with $2, 3$, respectively. Then clearly,

$$\begin{aligned} b_n &= a_{n-1} + b_{n-1} + c_{n-1} \\ a_n &= a_{n-1} + b_{n-1} \\ c_n &= b_{n-1} + c_{n-1} \end{aligned}$$

so

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 2 | 4 | 9 | 21 | 50 | 120 |
| $b_n$ | 1 | 2 | 5 | 12 | 29 | 70 | 169 |
| $c_n$ | 0 | 1 | 3 | 8 | 20 | 49 | 119 |

and the answer is $\boxed{119}$.

**L3.** [**8**] If the rightmost digit of $L_2$ is $d$, then let $T = 4d - 1$ be the number of diagonals in a regular polygon. If Peter Pan chooses 2 of these diagonals at random (without replacement) and the probability that they have the same length is $\frac{m}{n}$ when expressed as a common fraction, find $m + n$.

*Proposed by Tushar Gona*

In this 10-gon, the distribution of diagonals with different lengths is $10, 10, 10, 5$. Then, the probability is

$$\frac{90 + 90 + 90 + 20}{1190} = \frac{29}{119},$$

so the answer is $\boxed{148}$.

**L4.** [**10**] Let

$$A = \frac{148}{r} + \frac{148}{r^2} + \frac{148}{r^3} + \cdots, \quad B = \frac{1}{r^2} + \frac{2}{r^4} + \frac{3}{r^6} + \cdots,$$

where $r$ is a real number satisfying $|r| > 1$. Given that $A^2 = B$, the value of $|r|$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers $m, n$. Compute $m + n$.

*Proposed by Aaron Hu*

We have that $A = \frac{\frac{148}{r}}{1-\frac{1}{r}} = \frac{148}{r-1}$, and

$$B = \left(\frac{1}{r^2} + \frac{1}{r^4} + \frac{1}{r^6} + \cdots\right)\left(1 + \frac{1}{r^2} + \frac{1}{r^4} + \cdots\right) = \frac{1}{r^2-1} \cdot \frac{r^2}{r^2-1} = \frac{r^2}{(r^2-1)^2}.$$

Then $A^2 = B$ is equivalent to

$$\frac{148^2}{(r-1)^2} = \frac{r^2}{(r^2-1)^2} \implies r = 148(r+1) \implies r = -\frac{148}{147},$$

so the answer is $148 + 147 = \boxed{295}$.

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**S1.** **[5]** If $a^2 + a + 1 = 0$, evaluate $f(a)$ for $f(x) = x^{37} + 2x^2 + x$.

*Proposed by Sharvaa Selvan*

The solutions to $a^2 + a + 1 = 0$ are $e^{\pm 2\pi i/3}$. Therefore, taking $a = e^{2\pi i/3}$, $x^{37} = x = e^{2\pi i/3}$ while $x^2 = e^{4\pi i/3}$. Therefore, in $f(a)$, the imaginary parts cancel and we are left with $4 \cdot \frac{-1}{2} = \boxed{-2}$.

**S2.** **[6]** In an Asian household, there are two distinct types of chopsticks, wood and bamboo. Only chopsticks made of different materials are distinguishable. The youngest in the family takes 8 chopsticks uniformly at random. The probability that $|2 \cdot S_1|$ pairs, each pair with two identical chopsticks, can be made is $\frac{4}{17}$. The minimum possible number of total chopsticks is $M$. If there are $M$ chopsticks, the minimum number of bamboo chopsticks is $B$ and the number of wooden chopsticks is $W = M - B$. Find the product $BMW$.

*Proposed by Alex Li*

Clearly, $m$ must be divisible by 17 since the denominator of the probability is a factor of $m$ choose 8. The fraction is also relatively clean, hinting that the binomials that appear in the numerator could be 16 choose something or 15 choose something. In fact,

$$\frac{\binom{2}{0}\binom{15}{8} + \binom{2}{2}\binom{15}{6}}{\binom{17}{8}} = \frac{4}{17},$$

so 17 does indeed work. We can check that $w = 16$ gives probability $\frac{9}{17}$. Therefore, the answer is $2 \cdot 17 \cdot 15 = \boxed{510}$.

**S3.** **[6]** The distinct prime divisors of 510 are $p_1, p_2, \ldots, p_k$. Let $S$ be the set of all positive integers of the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where all $e_i$ are nonnegative integers. If

$$\sum_{x \in S} \frac{1}{\phi(x)} = \frac{m}{n}$$

when expressed as a common fraction, find the sum of the (not necessarily distinct) prime divisors of $m$. (For a positive integer $x$, $\phi(x)$ is the number of positive integers less than or equal to and relatively prime to $x$.)

*Proposed by Alex Li*

The key to this is that $\phi(n)$ is multiplicative, meaning we can split the sum into the product of three sums. That is,

$$\sum_{n\in S}\frac{1}{\phi(n)} = \left(\frac{1}{\phi(1)}+\frac{1}{\phi(2)}+\frac{1}{\phi(4)}+\cdots\right)\left(\frac{1}{\phi(1)}+\frac{1}{\phi(3)}+\frac{1}{\phi(9)}+\cdots\right)$$
$$\left(\frac{1}{\phi(1)}+\frac{1}{\phi(5)}+\frac{1}{\phi(25)}+\cdots\right)\left(\frac{1}{\phi(1)}+\frac{1}{\phi(17)}+\frac{1}{\phi(289)}+\cdots\right).$$

$\phi(p^k) = (p-1)p^{k-1}$ so the first factor on the RHS is

$$1+1+\frac{1}{2}+\frac{1}{4}+\cdots = 3.$$

The second factor is

$$1+\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\cdots = \frac{7}{4}.$$

The third factor is

$$1+\frac{1}{4}+\frac{1}{20}+\frac{1}{100}+\cdots = \frac{21}{16},$$

and the fourth factor is

$$1+\frac{1}{16}+\frac{1}{272}+\frac{1}{4624}+\cdots = \frac{273}{256},$$

so the total sum is

$$3\cdot\frac{7}{4}\cdot\frac{21}{16}\cdot\frac{273}{256},$$

giving $m = 21^2\cdot 273 = 3^3\cdot 7^3\cdot 13$, so the answer is $\boxed{43}$.

**S4.** **[8]** Charley gives Aaron the equation $x^3+S_3x^2+S_3x+S_3=0$. After Aaron solves it, he finds that one of the real roots is within 0.001 to some integer. What is the absolute value of this integer?

*Proposed by Alex Li*

Note that the solutions to $x^2+43x+43$ are $\frac{-43\pm\sqrt{43\cdot 39}}{2}$, which gives approximately $-42$ and $-1$. If we plug $x=-1$ into $x^3+43x^2+43x+43$, we find the value 42. Plugging $x=-42$ in gives $-42^3+43\cdot 42^2-43\cdot 42+43=1$, which is relatively close to 0. From this, we know the answer is $\boxed{42}$, where the integer was $-42$. In fact, some use of Wolfram Alpha seems to give 1 for the polynomial with any degree $n>1$.

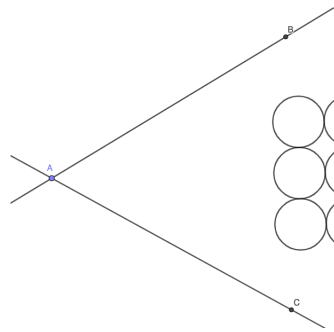
**S5.** **[9]** How many (non-degenerate) triangles with positive integer sidelengths $a, b, c$ satisfy

$$(a^2+S_4)(b^2+S_4) = c^2?$$

*Proposed by Alex Li*

Another troll: The RHS can be expressed as $a^2+b^2-2ab\cos\theta$. Note that $a^2+b^2\geq 2ab$ by AM-GM. However, the LHS is gigantic compared to this, with at least a value of $42a^2+42b^2$, so there are $\boxed{0}$ triangles that satisfy this

**S6.** **[10]** $N^2$ Survey Corps scouts rush at the Beast Titan with constant speed $5+S_5$ meters per second. Model each Corps scout as a circle with radius 1 meter, where the centermost circle's center in the front line is initially, at $t=0$, 501 meters away from the Beast Titan. In their square formation of $N\times N$, where $N$ is an odd positive integer, the circles are tangent to each other.




Every second, with the first throw at $t = 0$, the Beast Titan (modelled as a stationary point) throws rocks in a $60^\circ$ angle shown above, with the angle symmetric about the horizontal symmetry axis of the formation. Each throw is 1 rock thick, with each rock instantaneously taking out exactly one scout in the front line (given that part of the scout's circle is in the angle) after it is thrown. The back lines are unaffected. The minimum $N$ where at least 1 scout evades all rock throws is $n$. Find the nearest multiple of 50 to $n$.

*Proposed by Alex Li*

It is clear that the minimum $N$ must occur when only 2 scouts from the last row make it past the angle's rays, and everyone else is taken out. When these 2 people make it past the rays, they must either be tangent to the rays or to the left of the tangents.

We can find that, at the beginning, the center of the scout (call him Levi) at the upper right corner of the formation is $(498 + 2N, N - 1)$, where $(0, 0)$ is taken to be the Beast Titan's position. Since Levi moves horizontally, his center's $y$ position never changes, and the tangent point to the ray is found to be $(\sqrt{3}(N - 1 - \sqrt{3}), N - 1 - \sqrt{3})$. This implies that Levi's ending position is

$$\left((N-1)\sqrt{3} - \frac{7}{2}, N - 1\right).$$

To get here, Levi would have moved at least

$$498 + 2N - (N-1)\sqrt{3} + \frac{7}{2} = \frac{1003}{2} + \sqrt{3} + N(2 - \sqrt{3})$$

meters. This requires time

$$\Delta t = \frac{1003}{10} + \frac{\sqrt{3}}{5} + \frac{N(2-\sqrt{3})}{5}.$$

Since the throws start at $t = 0$, $\lfloor \Delta t \rfloor \approx N$. We want an approximate value of $N$, so we equate these

$$N = \frac{1003}{10} + \frac{\sqrt{3}}{5} + \frac{N(2-\sqrt{3})}{5}$$

$$\frac{3+\sqrt{3}}{5} N = \frac{1003 + 2\sqrt{3}}{10}$$

$$N = \frac{1003 + 2\sqrt{3}}{2(3+\sqrt{3})}.$$

Rationalizing, $N = (3003 - 997\sqrt{3})/12$. This is approximately $250 - 1730/12 \approx 250 - 144 = 106$, so the answer is $\boxed{100}$.

**A1.** **[5]** The number $RICKANDMORTY$ has each distinct letter represent a (not-necessarily distinct) digit in base 16, with $R > 0$. If the sum of all possible values of $RICKANDMORTY$ is $x$, find the largest integer $n$ such that $2^n \mid x$.

*Proposed by Alex Li*

$R$ appears twice at the positions $16^{11}$ and $16^2$, and everything else appears once. For a fixed $R$, the other 10 letters have $16^{10}$ choices. Over all $RICKANDMORTY$'s, the contributions to the sum from $R$'s is then

$$16^{10}(16^{11} + 16^2)(1 + 2 + 3 + \cdots + 15).$$

Consider $I$ at position $16^{10}$. A fixed $I$ gives the other 11 letters $15 \cdot 16^9$ choices, so the contributions from $I$'s is

$$15 \cdot 16^{19}(1 + 2 + 3 + \cdots + 15).$$

We can similarly add for the others to find a total sum of

$$(1 + 2 + \cdots + 15)(16^{21} + 16^{12} + 15(16^{19} + 16^{18} + \cdots + 16^{12} + 16^{10} + 16^9)).$$

The first factor is 120, giving 3 powers of 2 while the second gives $4 \cdot 9 = 36$ powers of 2. Therefore, $n = \boxed{39}$.

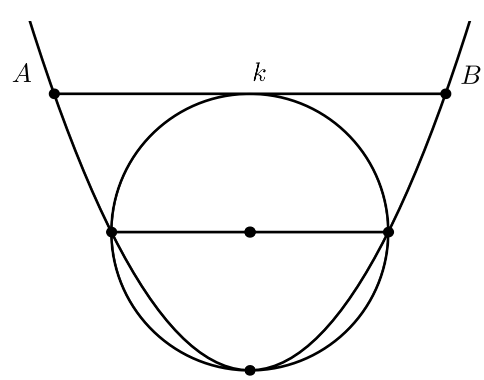
**A2.** **[7]** Rick decides to send Morty to the school called Amongus Academy. On day 1, Morty needs to travel from point $(0,0)$ to $(\frac{A_1}{3}, \frac{A_1}{3})$. If Morty must do a task at point $(6,9)$ and he moves 1 unit rightwards or upwards at a time, find the number of paths Morty can take, modulo 61.

*Proposed by Sharvaa Selvan*

We want $\binom{15}{6}\binom{11}{7} \equiv 5 \cdot 7 \cdot 11 \cdot 13 \cdot 11 \cdot 10 \cdot 3 \pmod{61}$

$$\begin{aligned} &\equiv_{61} 5 \cdot 16 \cdot 13 \cdot 11 \cdot 10 \cdot 3 \\ &\equiv_{61} 19 \cdot 13 \cdot 11 \cdot 10 \cdot 3 \\ &\equiv_{61} -4 \cdot 13 \cdot 11 \cdot 10 \\ &\equiv_{61} 9 \cdot 10 \cdot 11 \\ &\equiv_{61} 9 \cdot -12 \\ &\equiv_{61} \boxed{14}. \end{aligned}$$

**A3.** **[9]** A parabola and circle of radius $A_2$ are shown below, with $k = AB$. Find $\lfloor k \rfloor$. (That is, find the least integer greater than or equal to $k$.)



*Proposed by Alex Li*

Let the vertex be at $(0,0)$, so the parabola goes through $(14,14)$ and $(-14,14)$. The equation for the parabola is then $y = \frac{1}{14}x^2$. At $y = 28$, $x = \pm 14\sqrt{2}$, so $k = 28\sqrt{2}$. Using $\sqrt{2} \approx 1.41$ gives $\lfloor k \rfloor = \boxed{39}$.

**A4.** [**10**] Let $d$ be the units digit of $A_2$. Ritvik takes an isosceles trapezoid $ABCD$ with longer base of length $CD = A_3$ and legs $BC, AD$ of length $d$. Tushar measures an angle of the trapezoid to be $60°$. He then places two circles of equal radius, with $\omega_D$ inscribed in $\angle ADC$ and $\omega_C$ inscribed in $\angle BCD$. Ritvik takes another two circles of equal radius to the original two and inscribes $\omega_A$ in $\angle DAB$ and $\omega_B$ in $\angle CBA$. Now, $\omega_A$ is tangent to $\omega_D$ at $X$ and $\omega_B$ is tangent to $\omega_C$ at $Y$. Then, $XY = \sqrt{a} + \sqrt{b}$ for positive integers $a$ and $b$. Find $a + b$.

*Proposed by Alex Li*

Dropping perpendiculars from $A$ and $B$ onto $CD$ reveals that $AB = 31$. Furthermore, the line joining the centers of $\omega_A$ and $\omega_D$ is parallel to $AD$, so the height is $r + r\sqrt{3} + r = r(2+\sqrt{3}) = 4\sqrt{3}$, giving common radius $r = 8\sqrt{3} - 12$.

$XY$ is clearly on the median of $ABCD$, so we want $XM$, where $M$ is the midpoint of $AD$. A line through the center of $\omega_A$ parallel to the bases reveals $XM = \frac{2r}{\sqrt{3}}$, so

$$XY = \frac{31+39}{2} - \frac{4(8\sqrt{3}-12)}{\sqrt{3}} = 3 + 16\sqrt{3} = \sqrt{9} + \sqrt{768},$$

so the answer is $\boxed{777}$.

**A5.** [**12**] Find the sum of all real values of $x$ in the interval $[1, A_4]$ such that $1, \{x\}$, and $\{x^2\}$ form a geometric progression with a non-zero ratio, where $\{x\}$ denotes the fractional part of $x$.

*Proposed by Ritvik Teegavarapu*

Let $x = n + y$ where $n$ is a positive integer and $0 < y < 1$. Then, $1, y, y^2$ is the geometric progression. Furthermore, $x^2 = n^2 + 2ny + y^2$ must yield that $2ny$ is an integer. Working through each $n$:

$$n = 1 \implies y = \frac{1}{2}$$
$$n = 2 \implies y = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$$
$$n = 3 \implies y = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}.$$

Now, the partial sums from these $n$ are $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$. Therefore, for $n$ from 1 to 776, we have a total sum of

$$\frac{1+3+5+\cdots+1551}{2} = \frac{776^2}{2} = \boxed{301088}.$$

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**M1.** [**7**] Triangle $ABC$ has side lengths $AB = 13, BC = 14$, and $CA = 15$. A circle $\omega$ is drawn tangent to both $BC$ at its midpoint and the circumcircle of $ABC$. If the product of all possible values of the radius of $\omega$ is $t$, find $4t$.

*Proposed by Alex Li*

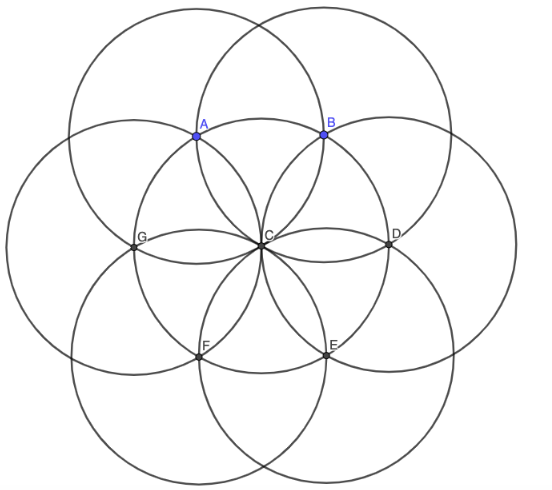
Note that the perpendicular from $O$ to $BC$ has length $\frac{33}{8}$, so the radius $r_1$ of the smaller $\omega$ is $\frac{65-33}{2\cdot 8} = 2$.

On the other hand, the radius $r_2$ of the larger $\omega$ must be $\frac{65-2\cdot 8}{8} = \frac{49}{8}$.

Therefore, the answer is $\boxed{49}$.

**M2.** **[8]** Aaron is pelting a watermelon at the ground. The watermelon has a splash zone of radius $M_1$. 6 people are standing at the vertices of a hexagon of side length $M_1$. If Aaron's watermelon lands uniformly at random inside the hexagon and the probability at least 3 people are in the splash zone is $r\sqrt{t}\pi - s$ for rational $r$ and positive integers $t$ and $s$, with $t$ not divisible by the square of any prime, find $81r + 9t + s$.

*Proposed by Alex Li*




The hexagon is $ABDEFG$ and the areas are the smaller sectors formed by the intersection of circles. They each have area

$$2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}.$$

There are 6 of them, so the total area is $6 \cdot \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$. The area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4}$, giving a probability of

$$\frac{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{4}} = \frac{4\sqrt{3}}{9}\pi - 2,$$

so the answer is $\boxed{65}$.

**M3.** **[9]** Three concentric circles have radii $1, 4$, and $9$, with each circle being the orbit of a planet. The period (time for complete revolution around the center) of each orbit satisfies Kepler's third law; the period is proportional to $r^{3/2}$, where $r$ is the radius of the orbit. If the period of the innermost orbit is 1 year, what is the minimal number of years it will take for all planets to be collinear again?

*Proposed by Alex Li*

The period of the innermost orbit is 1, so the periods of the other 2 are 8 and 27. In order to be collinear, they must be apart (in radians) by some multiple of $\pi$. After $t$ years, the second planet is at $2\pi t/8$ radians and the third is at $2\pi t/27$ radians. Therefore,

$$2\pi t \frac{19}{216} = \pi k_0 \implies k_0 = \frac{19t}{108}.$$

Similarly, the first planet is at $2\pi t$ radians, so

$$2\pi t \frac{7}{8} = \pi k_1 \implies k_1 = \frac{7t}{4}$$

$$2\pi t \frac{26}{27} = \pi k_2 \implies k_2 = \frac{52t}{27}.$$

All $k_i$ are integers, so the answer is $\boxed{108}$.

**M4.** **[12]** Let $T = \left\lfloor \frac{M_3}{10} \right\rfloor$. Determine the sum of the squares of all real solutions for the following equation, where $T$ is the coefficient of $\lfloor x \rfloor$:

$$x^2 - T\lfloor x \rfloor + \frac{51}{4} = 0.$$

*Proposed by Ritvik Teegavarapu*

If we ignore the floor function, the roots of the equation are at $x = 1.5$ and $8.5$. Therefore, we expect the valid $x$ to be near these roots.

Rewriting,

$$\lfloor x \rfloor = \frac{4x^2 + 51}{40}.$$

With some plugging in, we can find that $\lfloor x \rfloor = 2, 6, 7, 8$ all give solutions. The corresponding values of $x^2$ can be found to be $\frac{29}{4}$, $\frac{189}{4}$, $\frac{229}{4}$, and $\frac{269}{4}$, so the answer is $\boxed{179}$.

**M5.** **[15]** Vismay computes $T = \frac{M_4}{M_2}$. Karthik and Albert are given a polynomial $x^3 + 2\lceil T \rceil x^2 - 20x + 2\lfloor T \rfloor$ and told it has roots $pq, qr, rp$, where $p, q, r$ are complex numbers. Karthik then finds the monic cubic polynomial with roots $p, q, r$, while Albert finds the monic cubic polynomial with roots $-p, -q, -r$. They determine that there exists a unique positive real number $n$ such that when put into their polynomials, the outputs are equal. If $n^4 = \frac{a}{b}$ when expressed as a common fraction, find $10a + b$.

*Proposed by Aaron Hu*

Let $P(x) = x^3 + ax^2 + bx + c$ and $Q(x) = x^3 - ax^2 + bx - c$ be Karthik's and Albert's polynomials, respectively, which is possible from Vieta.

Therefore,

$$pq + qr + rp = b,$$
$$p^2qr + pq^2r + pqr^2 = pqr(p + q + r) = ac,$$
$$pqqrrp = p^2q^2r^2 = c^2,$$

so $(x - pq)(x - qr)(x - rp) = x^3 - bx^2 + acx - c^2$ from Vieta.

Comparing coefficients gives

$$(a, b, c) = (\pm 10i, -6, \pm 2i),$$

so

$$n^2 = -\tfrac{c}{a} = -\tfrac{1}{5} \implies n^4 = \frac{1}{25}$$

and the answer is $\boxed{35}$, as desired.