# Answer Key

1. 5
2. 35
3. 125
4. 5
5. 193
6. 50
7. 33
8. 520
9. 0
10. 376

**1.** A square piece of cookie dough has length 5. When a circular cookie cutter of radius $r$ is placed such that its center is the same as the center of the dough, pieces of dough are cut off from the corners of the square. Alex takes these pieces and uses them to fill in the spaces between the dough and cutter perfectly. Assuming the density and depth of the dough is constant, find $r\sqrt{\pi}$.

**Answer:** 5

*Proposed by Alex Li*

This is equivalent to the shapes of being the same area. Therefore, $25 = r^2\pi$ and $r = \boxed{5}$.

**2.** Albert tosses a fair coin and stops when he gets 2 heads in a row. If the probability that Albert stops after exactly 5 tosses is $\frac{m}{n}$ when expressed as a common fraction, find $m+n$.

**Answer:** 35

*Proposed by Sharvaa Selvan*

The third toss must be tails, and the last two tosses must both be heads. The first two tosses cannot be $HH$, which occurs with probability $\frac{3}{4}$. Then, the probability of ending after five tosses is $\frac{3}{32}$, so the answer is $\boxed{35}$.

**3.** Walter has a rectangular piece of paper $ABCD$. He folds the paper along $\overline{AE}$, where $E$ is a point on $\overline{CD}$ satisfying $EC = 3$, $ED = 5$. Given that $D$ lies on $\overline{BC}$ after the fold, compute $AE^2$.

**Answer:** 125

*Proposed by Aaron Hu*

Let $D'$ be the image of $D$ after the fold. Then $\angle AD'E = 90^\circ$, so $\triangle ABD' \sim \triangle D'CE$. Noting that $AB = 8$ and $\triangle D'CE$ is a $3-4-5$ triangle gives $AD = AD' = 10$, so

$$AE^2 = AD^2 + DE^2 = 10^2 + 5^2 = \boxed{125},$$

as desired.

**4.** The *digit sum* of a number can be found by repeatedly taking the sum of digits of the number until a single digit is found. For example, the digit sum of 369 is found as follows: Add the digits to get $3+6+9 = 18$, then add the resulting number's digits to get $1+8 = 9$ as the digit sum. Find the digit sum of $2^{2021}$.

**Answer:** 5

*Proposed by Tushar Gona*

This is the same as taking $\pmod 9$. Since $2^6 \equiv 1 \pmod 9$, $2^{2021} \equiv 2^5 \equiv \boxed{5} \pmod 9$.

**5.** Pam has made a regular 2022-gon using gumdrops and toothpicks. She chooses 6 distinct vertices. If the expected number of equilateral triangles that has all vertices among these 6 is $\frac{m}{n}$ when expressed as a common fraction, find the sum of the distinct prime factors of $mn$.

**Answer:** 193

*Proposed by Alex Li*

Note that the only possible equilateral triangles are the $\frac{2022}{3} = 674$ with each side cutting the 2022 sides to 674 sides and 1348 sides. Any smaller will have 1 vertex strictly inside or strictly outside the 2022-gon.

The probability that there is exactly 1 equilateral triangle is

$$\frac{\binom{674}{1} \cdot \left(\binom{2019}{3} - \binom{673}{1}\right)}{\binom{2022}{6}}.$$

The probability that there are exactly 2 equilateral triangles is

$$\frac{\binom{674}{2}}{\binom{2022}{6}}.$$

Therefore, the expected value is

$$\frac{\binom{674}{1} \cdot \left(\binom{2019}{3} - \binom{673}{1}\right)}{\binom{2022}{6}} + \frac{2 \cdot \binom{674}{2}}{\binom{2022}{6}} = \frac{674 \cdot \binom{2019}{3}}{\binom{2022}{6}},$$

which we can simplify to get $\frac{2}{2021 \cdot 101}$, so the answer is $2 + 43 + 47 + 101 = \boxed{193}$.

**6.** In a Mu Alpha Theta club, $p$ people are in the pre-calculus division and $c$ people are in the calculus division. If $p$ has 4 divisors and $c$ has 5 divisors, what is the least value of $p + c$ that has 6 divisors? Here, a divisor is a positive integer divisor.

**Answer:** 50

*Proposed by Alex Li*

Clearly, the least value of $c$ is 16. Note $16 + 8 = 24$ has more than 6 divisors and $16 + 27 = 43$ has 2 divisors. Therefore, we will try finding $p + c$ where $c = 2q$ for odd prime $q$. Adding,

$$p + c = 16 + 2q = 2(q + 8).$$

In order to have 6 divisors, it could be of the forms, $p_0^5, p_0^2 p_1$. The first one is impossible since $p + c$ has only 1 factor of 2. For the second one, we let $p_1 = 2$ to get:

$$2(8 + q) = 2p_0^2 \implies 8 + q = p_0^2,$$

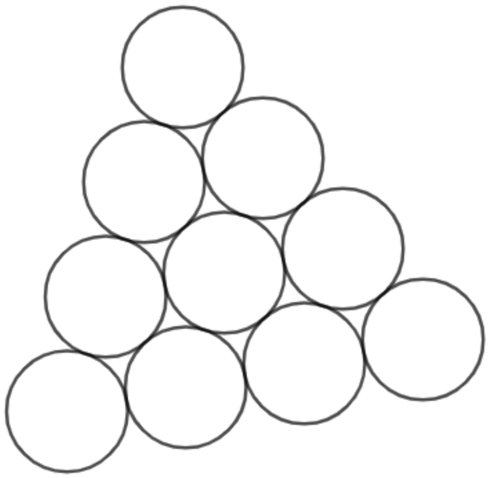
and the least allowed $p_0^2$ is 25, giving $q = 17$. Then, $p + c = \boxed{50}$. This is less than 81, so we know we have used the least possible value of $c$.

**7.** Inside an equilateral triangle $ABC$, a smaller equilateral triangle $XYZ$ with side length 2 is placed with the opposite orientation such that $X$ is closest to $BC$, $Y$ is closest to $AC$, and $Z$ is closest to $AB$. The distance from $X$ to $BC$ is $\sqrt{3} + 1$, from $Y$ to $AC$ is 1, and from $Z$ to $AB$ is 1. If the area of $ABC$ is $a\sqrt{b} + c$ for positive integers $a, b, c$ and $b$ is not divisible by the square of any prime, find $a + b + c$.

**Answer:** 33

*Proposed by Alex Li*

Fill triangle $ABC$ with 10 circles of radius 1 so that they are layered:



(Imagine there is a triangle circumscribing the whole figure.)

Then, the triangle $XYZ$ has each vertex at one of the centers of these circles. In particular $X$ is at the center of the figure, so $\sqrt{3}+1$ is $1/3$ of triangle $ABC$'s height. That is $h = 3\sqrt{3}+3$ and $AB = 6+2\sqrt{3}$, giving an area of $\frac{\sqrt{3}}{4}(48+24\sqrt{3}) = 12\sqrt{3}+18$. The answer is then $12+3+18 = \boxed{33}$.

**8.** Page 23 of a notebook has 10 lines. There are 4 distinct names written on the lines, with each line containing exactly 1 name. If each name appears at least once, and the number of ways to fill all 10 lines is $N$, find $N \pmod{1000}$.

Since the lines are on a page, they can be distinguished form each other.

**Answer:** 520

*Proposed by Alex Li*

We will solve this using PIE.

Without regards to the restrictions, there are $4^{10}$ ways to place names onto the page.

Now, we find the number of ways to place names such that at least one name does not appear. There are $4 \cdot 3^{10}$ ways to place 3 of the names while keeping the fourth name out. This overcounts placing 2 of the names while keeping the third and fourth out, which is $6 \cdot 2^{10}$. We need to add back $4 \cdot 1^{10}$ because we counted the case where only one name appears $3 - 3 = 0$ times from the above. Therefore, there are

$$4 \cdot 3^{10} - 6 \cdot 2^{10} + 4 \cdot 1^{10}$$

ways to make sure at least one name does not appear.

The desired number is

$$4^{10} - 4 \cdot 3^{10} + 6 \cdot 2^{10} - 4 \cdot 1^{10} \equiv 576 - 196 + 144 - 4 \equiv \boxed{520} \pmod{1000}.$$

**9.** Find the number of ordered pairs of positive integers $(x, y)$ such that

$$y(x^2 + 3y^2) = 12x^3,$$

and $x + y < 2021$.

**Answer:** 505

*Proposed by Aaron Hu*

Multiplying both sides by 18 and rearranging gives

$$(x+3y)^3 = (x-3y)^3 + (6x)^3,$$

and since $x, y$ are positive, we must have from Fermat's Last Theorem that $x = 3y$. However, plugging this back into the given equation yields

$$y(9y^2 + 3y^2) = 12y^3 = 12 \cdot 27y^3.$$

Unless $y = 0$, this equation is never satisfied. However, $y$ is a positive integer, so the answer is $\boxed{0}$.

**10.** TThe sum

$$\sum_{n=1}^{148} \left\lfloor \frac{2^n}{3} \right\rfloor$$

can be expressed as $\frac{2^a - c}{b}$ for positive integers $a, b, c$, with $c$ minimized. Compute $a + b + c$.

**Answer:** 376

*Proposed by Ritvik Teegaravapu*

For even $n$, $2^n \equiv 1 \pmod 3$. For odd $n$, $2^n \equiv 2 \pmod 3$, so the total sum is

$$\frac{2^1 + 2^3 + \cdots + 2^{147} - 2 \cdot 74}{3} + \frac{2^2 + 2^4 + \cdots + 2^{148} - 74}{3}.$$

This can be simplified to

$$\frac{2^{149} - 2 - 222}{3} = \frac{2^{149} - 224}{3},$$

so the answer is $149 + 224 + 3 = \boxed{376}$.