Florida Student Association of Mathematics

2021-2022 Everything Tryout

January 16-23, 2022

Round 3: Combinatorics

1. The integers $1, 2, 3, 4, 5, 6, 7$ are written on a blackboard. Every second, a uniformly random pair of distinct numbers are erased and replaced with their maximum or minimum with equal probability. The probability that the last number remaining after six seconds is 3 can be expressed as a simplified fraction $\frac{m}{n}$. Compute $m+n$.

2. When drawing a hand of five cards from a perfectly shuffled standard deck of 52 playing cards, the probability of a royal flush (drawing five cards of the same suit with one each of the ranks $10, J, Q, K, A$) is p. However, if I remove a card from the deck, the probability of drawing a royal flush from the 51 remaining cards is now either $p+q$ or $p-r$ for some positive reals p and q. If the ratio q/r can be expressed as a simplified fraction m/n, find m^2+n.

3. Aaron and Alex each take the same 10-question short answer test. After the test, Albert tells them Aaron got 9 correct and Alex got 6 correct. However, they do not know which questions they got correct, so any possible subset of correct answers is equally likely. If they realize their answers to question 1 were different, the probability Aaron answered question 1 correctly can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$. Note: If both Aaron and Alex are wrong on a question, there is 0 probability of them getting the same answer.

4. Aaron lives in a cube of side length 2. A light bulb is placed at each vertex of the cube. Each light bulb can either be turned on or off. However, the light bulbs are unstable, and if two light bulbs are apart by a distance of 2 are on simultaneously, both lights explode and so will Aaron. How many possible configurations of on/off light bulbs exist if Aaron does not explode?

5. There are N ways to erase some (possibly none but not all) digits from the number

$$111222333444555666777888999$$

so that concatenating the remaining digits gives a multiple of 3. When N is written in binary, how many 1s are needed? *Consider all digits to be distinguishable; that is, count erasing the first* 3 *and erasing the second* 3 *as different ways.*

6. 20 dots are in a circle, as shown below. How many ways are there to draw 10 line segments such that every dot meets exactly one segment, and every segment intersects exactly 1 other? Two configurations are different if and only if some dot d is connected to e in one and f in the other for $e \neq f$.