Algebra and Number Theory

1. Suppose positive integers a, b satisfy $2 \leq a, b \leq 100$. There are exactly 2 positive real solutions of x in the equation

$$\log_{2^a} \log_{2^b} x = \log_{2^{b+1}} \log_{2^{a-1}} x.$$

Find the greatest possible value of ab.

Proposed by Alex Li.

Solution: Let $x = 2^y$. Note that y can now take any real value. Then, the equation reduces to

$$\frac{1}{a} \log_2 \frac{y}{b} = \frac{1}{b+1} \log_2 \frac{y}{a-1}.$$

Taking 2 to the power of each side,

$$\left(\frac{y}{b}\right)^{1/a} = \left(\frac{y}{a-1}\right)^{1/(b+1)} \implies y^{b+1-a} = \frac{b^{b+1}}{(a-1)^a}.$$

For this to have exactly two real solutions, we need $2 \mid b + 1 - a$, and we can easily see that $b = 100, a = 99$ works, giving an answer of $\boxed{9900}$.

2. Compute the minimum positive integer n which satisfies the following inequality:

$$\frac{(1 + 2 + \cdots + n)(1^3 + 2^3 + \cdots + n^3)}{(1^2 + 2^2 + \cdots + n^2)^2} > 1.124$$

Proposed by Karthik Vedula.

Solution: Using

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

and

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

the LHS of the given inequality becomes

$$\frac{9}{2} \cdot \frac{n(n+1)}{(2n+1)^2} > 1.124.$$

Multiplying out the denominator and moving terms yields

$$n^2 + n > 281,$$

so the minimum n is $\boxed{17}$.

3. For a positive integer n, consider variables $a_1, a_2, \cdots a_n$, and the expression $(\pm a_1 \pm a_2 \pm \cdots \pm a_n)^2$, where each $\pm$ sign has an equal probability of representing a plus or a minus. Let $f(n)$ denote the expected value of the number of terms of the expanded expression (with all like terms combined) with a positive coefficient. Compute the least positive integer n such that $f(n)$ is an integer, and it is a multiple of 20.

Proposed by Karthik Vedula.

Solution: First, we compute a closed form for $f(n)$. Suppose that out of the n signs, k of them were positive, and $n-k$ were negative. Note that all a_i^2 terms will have positive integers, and all $\pm 2a_ia_j$ will be positive iff a_i and a_j had the same sign. This means that there will be $n+\binom{k}{2}+\binom{n-k}{2}$ positive terms. The scenario of k positives occurs with $\frac{\binom{n}{k}}{2^n}$ probability, so the desired expected value is

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^n}\left(n+\binom{k}{2}+\binom{n-k}{2}\right)=n+\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\binom{k}{2}+\binom{n-k}{2}\right)$$

Note that

$$\binom{k}{2}+\binom{n-k}{2}=\frac{k(k-1)+(n-k)(n-k-1)}{2}=\frac{n^2-n-2nk+2k^2}{2}=-k(n-k)+\frac{n^2-n}{2}$$

Applying this and taking out the $\frac{n^2-n}{2}$, we get

$$\frac{n^2+n}{2}-\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}k(n-k)=\frac{n^2+n}{2}-\frac{1}{2^n}\sum_{k=0}^{n}\binom{n-2}{k-1}n(n-1)=$$

$$=\frac{n^2+n}{2}-\frac{1}{2^n}\cdot 2^{n-2}n(n-1)=\frac{n^2+n}{2}-\frac{n^2-n}{4}=\frac{n^2+3n}{4}$$

Now, if 20 divides $f(n)$, 80 must divide $n(n+3)$. Note that either n or $n+3$ is a multiple of 5:

- n is a multiple of 5: Testing gives $45\cdot 48=80\cdot 27$ as the smallest in this case.
- $n+3$ is a multiple of 5: Testing gives $32\cdot 35=80\cdot 14$ as the smallest in this case.

The answer is therefore $\boxed{32}$.

4. Find the largest positive integer n such that there exist distinct positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_1a_2\cdots a_n=2021^{2021}.$$

Proposed by Aaron Hu.

Solution: Let $f(n)$ denote the number of not necessarily distinct prime factors of positive integer n. Then for integer $0\le k\le 2021$, there are $k+1$ integers n satisfying $n\mid 2021^{2021}$ and $f(n)=k$, namely, all numbers of the form

$$43^i47^{k-i},\quad 0\le i\le k.$$

Note that

$$\sum_{i=1}^{n}i(i+1)=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{3},$$

so the summation for $n=22$ is equal to 4048. This gives the following as the optimal method of selecting a_i :

- For $0\le k<22$, choose the $k+1$ numbers n satisfying $n\mid 2021^{2021}$ and $f(n)=k$. This gives 3542 prime factors.
- Choose 21 numbers n satisfying $n\mid 2021^{2021}$ and $f(n)=22$. This gives 462 prime factors.
- Choose n satisfying $n\mid 2021^{2021}$ and $f(n)=38$, for 38 prime factors.

This works as it gives a total of $3542+462+38=4042$ prime factors. This gives a total of

$$(1+2+\ldots+22)+21+1=\boxed{275}$$

numbers, as desired.

5. Suppose that a, b are positive integers such that $\sqrt{2^{76} - 2^{51} - 2^{50} + 1} = a\sqrt{b}$, and b is not divisible by the square of any prime. b is the product of a 1-digit prime, a 2-digit prime, a 3-digit prime, and a 4-digit prime. Compute this 4-digit prime number.

Proposed by Karthik Vedula.

Solution: Let $x = 2^{25}$, so the term inside the radical is

$$2x^3 - 3x^2 + 1 = (x-1)^2(2x+1),$$

so b is the non-square part of $2^{26} + 1$. From here, after setting $x = 2^6 = 64$, we can use Sophie-Germain on

$$1^4 + 4 \cdot x^4 = (1 + 2x + 2x^2)(1 - 2x + 2x^2).$$

It is feasible to calculate these two factors as 8065 and 8321. Note that $8065 = 5 \cdot 1613$ and $8321 = 57 \cdot 157$, so the answer is $\boxed{1613}$.

6. For a positive integer $k \geq 2$, let a_k, b_k and c_k be the complex roots of the equation below.

$$\left(x - \frac{1}{k-1}\right)\left(x - \frac{1}{k}\right)\left(x - \frac{1}{k+1}\right) = \frac{1}{k}$$

Given that the sum

$$\sum_{k=2}^{\infty} \frac{(a_k^2 + a_k)(b_k^2 + b_k)(c_k^2 + c_k)}{k+1}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, determine $m + n$.

Proposed by Ritvik Teegavarapu.

Solution: This is essentially two problems in one: the first part is finding $a_k b_k c_k$ and $(1 + a_k)(1 + b_k)(1 + c_k)$, and the second part is a large partial fraction decomposition, followed by some simplification.

Write the given polynomial as

$$P(x) = \left(x - \frac{1}{k-1}\right)\left(x - \frac{1}{k}\right)\left(x - \frac{1}{k+1}\right) - \frac{1}{k} = 0.$$

Then, by Vieta's (and using a, b, c for clearness),

$$abc = \frac{1}{k(k-1)(k+1)} + \frac{1}{k} = \frac{k}{k^2 - 1}.$$

Note that

$$\begin{aligned}(1+a)(1+b)(1+c) &= -(-1-a)(-1-b)(-1-c) = -P(-1) \\ &= \frac{k}{k-1} \cdot \frac{k+1}{k} \cdot \frac{k+2}{k+1} + \frac{1}{k} \\ &= \frac{k^2 + 3k - 1}{k(k-1)}.\end{aligned}$$

Therefore, the term inside the desired summation is

$$\frac{k^2 + 3k - 1}{(k-1)^2(k+1)^2}.$$

To find the partial fraction decomposition, write the above as

$$\frac{A}{k-1} + \frac{B}{(k-1)^2} + \frac{C}{k+1} + \frac{D}{(k+1)^2},$$

and doing some expanding yields

$$\frac{k^2+3k-1}{(k-1)^2(k+1)^2} = \frac{1/2}{k-1} - \frac{1/2}{k+1} + \frac{3/4}{(k-1)^2} - \frac{3/4}{(k+1)^2}.$$

Clearly, many terms cancel in the summation, which leads to a final sum of

$$\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right) + \frac{3}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}\right) = \frac{27}{16},$$

so the answer is $\boxed{43}$.

Geometry

1. Triangle ABC, with side lengths $AB = 5, BC = 8$, and $CA = 7$ is circumscribed by circle ω. Another point A' on ω is chosen such that ABC and $A'BC$ have the same area. Find AA'.

Proposed by Alex Li.

Solution: $AA'BC$ is an isosceles trapezoid with $AB = A'C = 5$. Dropping an altitude AD from A to BC, D satisfies $BD = AB\cos 60^\circ = 5/2$ (note $\angle B = 60^\circ$ can be easily proven with Law of Cosines). Furthermore, an altitude $A'E$ from A' to BC satisfies $CE = 5/2$.

Therefore,

$$AA' = BC - BD - CE = 8 - 5 = \boxed{3}.$$

2. Consider an ellipse $\mathcal{E}$ with a horizontal major axis of length $2a$, where a is a positive real number. A circle Ω is inscribed tangent to the the endpoints of the vertical minor axis. Another circle, ω, of radius r, is externally tangent to ω and internally tangent to $\mathcal{E}$ at one endpoint of the major axis. Find the minimum possible value of the ratio a/r.

Proposed by Alex Li.

Solution: Note that the condition vital to this problem is that ω is tangent to $\mathcal{E}$.

Let the semiminor axis be b, so the radius of ω is $r = \frac{a-b}{2}$. The center of $\mathcal{E}$ can be set at $(0,0)$. The center of ω is at $\left(\frac{a+b}{2}, 0\right)$. Then, the equations of $\mathcal{E}$ and ω are, respectively,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

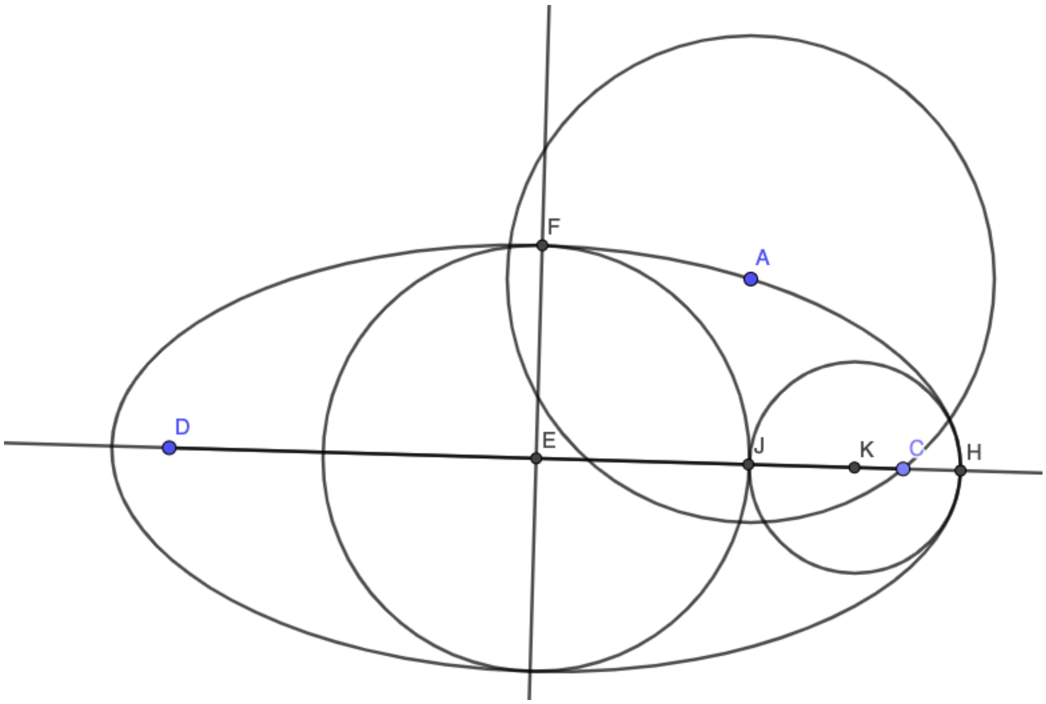
$$\left(x - \frac{a+b}{2}\right)^2 + y^2 = \left(\frac{a-b}{2}\right)^2.$$

After substituting in y^2 from the first equation into the second, we find

$$\left(1 - \frac{b^2}{a^2}\right) - (a+b)x + ab + b^2 = 0.$$

In order for the minimum $\frac{a}{r}$ to happen, there can be only one such solution x (namely, $x = a$). Using the discriminant of the above quadratic and setting to 0 will yield $a = 2b$. Therefore, the radius $r = \frac{b}{2} = \frac{a}{4}$ is maximum. Therefore, the answer is $\boxed{4}$.

Remark: Here is a diagram showing the result:



The circle centered at the point named A is to make sure AC remains the same for different b so that $DA + AC$ is constant is satisfied.

3. Consider $\triangle ABC$ with $AB = AC = 20\sqrt{21}$ and $BC = 40\sqrt{7}$. Let D, E denote the projections of A, B onto $\overline{BC}, \overline{CA}$, respectively. Let lines DE and AB intersect at P. Given that PD can be expressed as $a\sqrt{b}$ for positive integers a, b with square-free b, compute $a + b$.

Proposed by Aaron Hu.

Solution: Scale everything down by a factor of $20\sqrt{7}$. Then $AB = AC = \sqrt{3}$ and $BC = 2$. Note that $ABDE$ is cyclic, so from Power of a Point,

$$\sqrt{3} \cdot CE = CE \cdot CA = CD \cdot CB = 2 \implies CE = \frac{2\sqrt{3}}{3},$$

so $CE = \frac{2}{3}CA$. This implies that E is the centroid of $\triangle PBC$, and that P is the reflection of B over A. Then from Power of a Point again,

$$\frac{2}{3} \cdot PD^2 = PE \cdot PD = PA \cdot PB = 6 \implies PD = 3.$$

Scaling back up gives $PD = 60\sqrt{7}$, so the answer is $\boxed{67}$.

4. In acute triangle $\triangle ABC$, let H be its orthocenter (intersection of altitudes), and let O be its circumcenter (intersection of perpendicular bisectors of sides). Circle (AHO) is tangent to (ABC), $\tan A = \frac{26}{15}$, and $BC = 52$. Compute the area of $\triangle ABC$.

Proposed by Karthik Vedula.

Solution: The key idea is to note that $AH = 2OM$, where M is the midpoint of BC. To see this, note that $AH = AC \cos A \cdot \frac{1}{\sin B} = 2R \cos A$, so the result follows, where R is the circumradius. Using the condition that (AHO) is tangent to (ABC), we have $\angle AHO = 90°$, so the height to BC is $h = AH + OM = 3OM$.

Also, $R \sin A = 26$, so $R = \frac{26}{\sin A}$, giving $h = \frac{78 \cos A}{\sin A} = 45$. Therefore, the area is $\frac{52 \cdot 45}{2} = \boxed{1170}$.

5. Consider $\triangle ABC$ with $AB = 13$, $BC = 14$, $CA = 15$. Let M denote the midpoint of $\overline{AC}$, and let E be an variable point on $\overline{AB}$ such that $AE < EC$. Let D denote the point on $\overline{BC}$ such that $\overline{AD}$, $\overline{BM}$, $\overline{CE}$ concur and let F denote the point on $\overline{AC}$ such that $\overline{DF} \parallel \overline{AB}$. The maximum possible area of $\triangle EFM$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

Proposed by Aaron Hu.

Solution: Let $T = \overline{EM} \cap \overline{BC}$. Then since $\overline{AD}, \overline{BM}, \overline{CE}$ concur, we have

$$-1 = (T, D; B, C) \stackrel{E}{=} (M, \overline{DE} \cap \overline{AC}; A, C),$$

implying that $\overline{DE} \parallel \overline{AC}$. Then

$$\frac{AE}{AB} = \frac{FD}{AB} = \frac{CF}{CA} = r,$$

for $r \in [0, \frac{1}{2}]$. Then

$$[EFM] = [ABC] \cdot \frac{AE}{AB} \cdot \frac{FM}{AC} = 84r\left(\frac{1}{2} - r\right) \le \frac{21}{4},$$

where the last step follows from AM-GM. Equality holds at $r = \frac{1}{4}$, so the answer is $\boxed{2104}$.

6. Convex cyclic quadrilateral $APQB$ with vertices in that order is inscribed in circle ω with center O. There is a point T on segment $\overline{AB}$ such that $AT = AP$, $BT = BQ$, and the midpoint M of $\overline{PQ}$ satisfies $MT \perp AB$. Given $MO = 41$ and $MT = 40$, compute the sum of all possible values of TO.

Proposed by Albert Wang.

Solution: Let C be the projection of A onto PQ and D be the projection of B onto PQ. Let E be the reflection of P across C, and let F be the reflection of P across D.

Note that the power of M with respect to the circle centered on A passing through P equals the power of M with respect to the circle centered on B passing through Q. Therefore, since $MP \cdot ME = MQ \cdot MF$, $MP = MQ$, and C, D are the midpoints of $\overline{PE}, \overline{QF}$ respectively, we have $PC = PD$.

From this, the intersections P_1, P_2 of the line perpendicular to PQ through C with $(APQB)$ and the intersections Q_1, Q_2 of the line perpendicular to PQ through D with $(APQB)$ satisfy P_1, P_2, Q_1, Q_2 form a rectangle with sides parallel and perpendicular to PQ in some order.

Since $A \in \{P_1, P_2\}$ and $B \in \{Q_1, Q_2\}$ we have that either $PQ \parallel AB$ or $\overline{AB}$ is a diameter of $(APQB)$.

In the first case, we get TOM collinear, which means $TO = 81$ or $TO = 1$. The other case, O lies on AB and $\angle MTO = 90^\circ$, which gives $TO = 9$. So, $1 + 9 + 81 = \boxed{91}$ is our answer.

Combinatorics

1. The integers $1, 2, 3, 4, 5, 6, 7$ are written on a blackboard. Every second, a uniformly random pair of distinct numbers are erased and replaced with their maximum or minimum with equal probability. The probability that the last number remaining after six seconds is 3 can be expressed as a simplified fraction $\frac{m}{n}$. Compute $m + n$.

Proposed by Albert Wang.

Solution: Note that the operation is completely identical to randomly removing one of the two numbers, since $\{\min(a, b), \max(a, b)\} = \{a, b\}$. However, if the answer for i being the last number remaining is $f(i)$, then all of $f(1), f(2), \ldots, f(7)$ are equal, giving $\frac{1}{7} \to \boxed{8}$.

2. When drawing a hand of five cards from a perfectly shuffled standard deck of 52 playing cards, the probability of a royal flush (drawing five cards of the same suit with one each of the ranks $10, J, Q, K, A$) is p. However, if I remove a card from the deck, the probability of drawing a royal flush from the 51 remaining cards is now either $p + q$ or $p - r$ for some positive reals p and q. If the ratio q/r can be expressed as a simplified fraction m/n, find $m^2 + n$.

Proposed by Albert Wang.

Solution: Note that the case where the probability increases is when one of the 8 ranks not involved in the royal flush is chosen, and the case where the probability decreases is one of the 5 ranks from 10 through A are chosen. Note that the probability of drawing a royal flush after a random card is removed is still the same as it was before (instead of drawing from the top of the deck, put the top card at the bottom of the deck and then start drawing; by symmetry the probability is the same.) So, we can write

$$p = \frac{8}{13}(p + q) + \frac{5}{13}(p - r)$$

which implies $8q = 5r \to q/r = \frac{5}{8} \to \boxed{33}$.

3. Aaron and Alex each take the same 10-question short answer test. After the test, Albert tells them Aaron got 9 correct and Alex got 6 correct. However, they do not know which questions they got correct, so any possible subset of correct answers is equally likely. If they realize their answers to question 1 were different, the probability Aaron answered question 1 correctly can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$. Note: If both Aaron and Alex are wrong on a question, there is 0 probability of them getting the same answer.

Proposed by Alex Li.

Solution: There are three scenarios where they can get different answers to question 1: (1) both are wrong, (2) Aaron is right, Alex is wrong, and (3) Aaron is wrong, Alex is right.

(1): The probability Aaron gets it wrong is $\frac{\binom{9}{9}}{\binom{10}{9}}$, and the probability Alex gets it wrong is $\frac{\binom{9}{6}}{\binom{10}{6}}$. This gives probability

$$\frac{\binom{9}{9}}{\binom{10}{9}} \cdot \frac{\binom{9}{6}}{\binom{10}{6}}.$$

(2): Using the same procedure, the probability here is

$$\frac{\binom{9}{8}}{\binom{10}{9}} \cdot \frac{\binom{9}{6}}{\binom{10}{6}}.$$

(3): The probability here is

$$\frac{\binom{9}{9}}{\binom{10}{9}} \cdot \frac{\binom{9}{5}}{\binom{10}{6}}.$$

Now, we use conditional probability. Only the second case has Aaron getting the question right, so the desired probability is

$$\frac{\frac{\binom{9}{8}}{\binom{10}{9}} \cdot \frac{\binom{9}{6}}{\binom{10}{6}}}{\frac{\binom{9}{9}}{\binom{10}{9}} \cdot \frac{\binom{9}{6}}{\binom{10}{6}} + \frac{\binom{9}{8}}{\binom{10}{9}} \cdot \frac{\binom{9}{6}}{\binom{10}{6}} + \frac{\binom{9}{9}}{\binom{10}{9}} \cdot \frac{\binom{9}{5}}{\binom{10}{6}}} = \frac{18}{23},$$

giving an answer of $\boxed{41}$.

4. Aaron lives in a cube of side length 2. A light bulb is placed at each vertex of the cube. Each light bulb can either be turned on or off. However, the light bulbs are unstable, and if two light bulbs are apart by a distance of 2 are on simultaneously, both lights explode and so will Aaron. How many possible configurations of on/off light bulbs exist if Aaron does not explode? Rotations and reflections are considered distinct.

Proposed by Ritvik Teegavarapu.

Solution: Casework on the number of lights that are on:

- 0 lights on gives 1 way.
- 1 light on gives 8 ways.
- 2 lights on: If the two vertices are opposite, there are 4 ways. If they lie on the same face, it is 12 ways, so there are 16 in total.
- 3 lights on: It is easy to show that, if two of the vertices are on are opposite, then it is impossible. Therefore, the 3 lights form an equilateral triangle that "hugs" a vertex, giving 8.
- 4 lights on: This is 2.

Therefore, the answer is

$$1 + 8 + 16 + 8 + 2 = \boxed{35}.$$

5. There are N ways to erase some (possibly none but not all) digits from the number

$$111222333444555666777888999$$

so that concatenating the remaining digits gives a multiple of 3. When N is written in binary, how many 1s are needed? *Consider all digits to be distinguishable; that is, count erasing the first* 3 *and erasing the second* 3 *as different ways.*

Proposed by Albert Wang.

Solution: We can remove the $3, 6, 9$s however we want, giving 2^9 options for those digits. Reducing mod 3, we are just counting the number of ways to remove i $1, 4, 7$s and j $2, 5, 8$s where $i \equiv j \pmod 3$. Using the root of unity filter on

$$P(x) = \sum_{i=0}^{9} \sum_{j=0}^{9} \binom{9}{i}\binom{9}{j} x^{i+9-j}$$

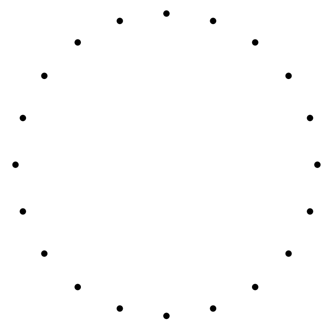
yields that there are $\lceil \frac{2^{18}}{3} \rceil$ ways to remove the non-multiples of 3. So, we have $N = 2^9 \lceil \frac{2^{18}}{3} \rceil - 1$. Note that in binary, $\lceil \frac{2^{18}}{3} \rceil$ is $\left(2^{16} + 2^{14} + \cdots + 2^2 + 2^0\right) + 1$, or

$$10101010101010110_2$$

which has 9 ones. Shifting it by 9 zeroes is the same as multiplication by 2^9, and subtracting one removes the ending 1 but gives 10 more at the end, giving $\boxed{18}$ ones:

$$N = 10101010101010101111111111_2.$$

6. 20 dots are in a circle, as shown below. How many ways are there to draw 10 line segments such that every dot meets exactly one segment, and every segment intersects exactly 1 other? Two configurations are different if and only if some dot d is connected to e in one and f in the other for $e \neq f$.



Proposed by Albert Wang.

Solution: Note that this is equivalent to partitioning the points into 5 quadrilaterals that do not intersect each other. Fix a point P and go around the circle, and let's say we keep a stack of the current regions we're placing points into, initially storing nothing. Now, we have two operations for each of the 20 dots we encounter:

- Place a dot in our current quadrilateral if it has less than 4 dots, and if it now has 4 dots then pop the stack and continue adding dots to the previous quadrilateral.
- Place a dot in a new quadrilateral and add it to the stack.

Note that the second move must be used 5 times and that the first move must be used 15 times. If we are now counting paths where each first move increments our y by 1 and our second move increments our x by 1, we are trying to move from $(0,0)$ to $(15,5)$ while always satisfying $y \geq x/3$. (Otherwise, if $y < x/3$ ever happens we will not have enough quadrilaterals to fit all of our dots.) This task is now easy to complete with a dynamic programming table (we go from $(15,5)$ to 0 instead for sake of easier computation):

x	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
			x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
						x	**4**	**9**	**15**	**22**	**30**	**39**	**49**	**60**	**72**	**85**
									x	**22**	**52**	**91**	**140**	**200**	**272**	**357**
												x	**140**	**340**	**612**	**969**
															x	$\boxed{\textbf{969}}$