# Florida Student Association of Mathematics

## 2022 ARML Tryout Solutions

April 2022

# Round 1

**Name:** ____________________

**1.** 49 **2.** 25

**1.** Let $N$ denote the starting page number, and suppose Alex reads $k$ pages. Then the last page he reads is $N+k-1$, so we have $\frac{1}{2}k(2N+k-1) = 1984$, implying that $k(2N+k-1) = 2^7 \times 31$. Note that $k < 2N+k-1$ and they are of opposite parity, which forces $k = 31, 2N+k-1 = 128$. This gives $N = \boxed{49}$, and we are done.

**2.** Drop a perpendicular from the center of the circle $O$ to a midpoint $M$ of a chord. Then, $OM = 3$ and $MA = 4$, where $A$ is one of the endpoints of the chord. Therefore, the radius is 5, and $a = \boxed{25}$.

# Round 2

**Name:** ______________________

**3.** 17 **4.** 18

**3.** The simplest sums are $1+p$ and $(1+q)(1+r)$ for primes $p,q,r$. If $q=2, r=3$, then we need 12 as the common sum, which easily gives $p=11$. Therefore, the answer is $6+11=\boxed{17}$.

**4.** Let $AD=x$ and $BD=y$. By Stewart's, $xy\cdot 4\sqrt{2}+4\sqrt{2}\cdot 9=16(x+y)=64\sqrt{2}$, so $xy=7$. Then, $x^2+y^2=(x+y)^2-2xy=\boxed{18}$.

# Round 3

**Name:** ______________________

**5.** 423 **6.** -4

**5.** Note $163681 = 4^2 \cdot 10^4 + 6^2 \cdot 10^2 + 9^2$, and completing the square gives $409^2 - 60^2 = 349 \cdot 469 = 349 \cdot 7 \cdot 67$, giving a sum of $\boxed{423}$.

**6.** Let $S_n$ be the sum of the products of roots taken $n$ at a time. (For example, $S_2 = ab + ac + ad + bc + bd + de$.)

By Newton's sums (and noting that $S_i = \pm 1$),

$$P_5 + P_4 + P_3 + P_2 + P_1 + 5 = 0 \implies P_5 + P_4 + P_2 + P_1 = -5 - P_3.$$

Note that $P_1 = -1$ by Vieta's and $P_2 = P_1 S_1 - 2S_2 = -1$. By Newton's sums,

$$P_3 + P_2 + P_1 + 3 = 0 \implies P_3 = -1,$$

so the answer is $\boxed{-4}$.

# Round 4

**Name:** ______________________

**7.** 4045 **8.** 33

**7.** Note

$$\frac{T_n}{T_n - 1} = \frac{\frac{n(n+1)}{2}}{\frac{(n-1)(n+2)}{2}} = \frac{n(n+1)}{(n-1)(n+2)}.$$

Therefore, the product telescopes to

$$P_{2022} = \frac{3}{1} \cdot \frac{2022}{2024} = \frac{3033}{1012} \implies \text{answer is } \boxed{4045}.$$

**8.** Let $f(x) = 2x^3 + 2x + 1$. Note that $43 \mid f(10) = 2021$, so for a solution $x$, we must have

$$43 \mid f(x) - f(10) = 2(x - 10)(x^2 + 10x + 101).$$

For $x \neq 10$, this is equivalent to $43 \mid x^2 + 10x + 101$, so

$$43 \mid x^2 + 10x + 101 - 7 \cdot 43 = x^2 + 10x - 200 = (x - 10)(x + 20).$$

Thus, the only other solution is $x = 23$, and the answer is $\boxed{33}$.

# Round 5

**Name:** ______________________

**9.** 1195 **10.** 24

**9.** Note that we can only have $d = 2, 3, 5$.

$d = 2$: $a, b, c \in \{2, 4, 6, \dots, 20\}$, so there are $\binom{10}{3}$ ways without considering other $d$. Note this counts $d = 4$ and 6. If $d = 4$, $a, b, c \in \{4, 8, 12, 16, 20\}$, which has $\binom{5}{3}$ ways. If $d = 6$, $a, b, c \in \{6, 12, 18\}$, which has 1 way. (Note $d = 4, 6$ cases have no overlap.) Therefore, $d = 2$ gives $\binom{10}{3} - \binom{5}{3} - 1 = 109$ ways.

$d = 3$: $a, b, c \in \{3, 6, 9, 12, 15, 18\}$, so $\binom{6}{3}$ ways without considering other $d$. If $d = 6$, there is 1 way. Thus, $d = 3$ gives $\binom{6}{3} - 1 = 19$ ways.

$d = 5$: $a, b, c \in \{5, 10, 15, 20\}$, which has $\binom{4}{3} = 4$ ways.

Therefore, the total number of ways to make $d$ prime is 132. There are $\binom{20}{3} = 1140$ ways in total, for an answer of $\frac{11}{95} \implies \boxed{1195}$.

**10.** Note that the values being shortest distances means they traverse along a great circle of the sphere. Since the diameter is at least $\pi$, the circumference of such great circles is $\pi^2 \approx 9$, so none of the given distances form an angle greater than $\pi$ radians with the center of the sphere.

Let $a, b, c$ be the angles (in radians) formed by the central angle from the center of $\Phi$ to the endpoints of $BC, CA$, and $AB$, respectively. Then, $a = \frac{3}{R}, b = \frac{4}{R}$, and $c = \frac{2}{R}$. By Law of Cosines and using that the line segments $AB, BC, CA$ form a right triangle, we have

$$2R^2 - 2R^2 \cos a + 2R^2 - 2R^2 \cos c = 2R^2 - 2R^2 \cos b \implies \cos \frac{3}{R} + \cos \frac{2}{R} = \cos \frac{4}{R} + 1.$$

Let $\cos \frac{1}{R} = x$. Then, the above equation implies

$$4x^3 - 3x + 2x^2 - 1 = 8x^4 - 8x^2 + 1 + 1 \implies 8x^4 - 4x^3 - 10x^2 + 3x + 3 = 0.$$

Note that $x = \sqrt{3}/2$ is a root to the above polynomial. Therefore, $\cos \frac{1}{R} = \cos \frac{\pi}{6}$, giving $R = \frac{6}{\pi}$, and the answer is $4\pi R = \boxed{24}$.

# Round 6

**Name:** ____________________

**11.** 8 **12.** 1793

**11.** Motivation: replace $(\sqrt{2})^x = a$ and a get the quartic $a^4 + a - 84 = 0$; rational root theorem works from here.

$x = 2\log_2 3$ and $y = 2\log_3 4 = 4\log_3 2$, so $xy = \boxed{8}$.

**12.**

# Round 7

**Name:** ______________________

**13.** 99 **14.** 1791

**13.**

**14.** It will help to visualize this problem on the portion of the unit lattice from $(100, 100)$ to $(999, 999)$. The primary claim is that the set of points forms a shortest taxicab path from $(100, 999)$ to $(999, 100)$, missing a point at each coincidence with $xy = 10^5$. There are only 8 such $x$-values in the range $(160, 200, 400, 800, 125, 250, 500, 625)$, the answer is easily seen to be eight less than the taxicab distance between the ends of the path plus one, giving $899 \times 2 - 8 + 1 = \boxed{1791}$. The proof of this claim is left as an exercise (consider the curves $xy = 10^5$, $(x + 1)(y + 1) = 10^5$.