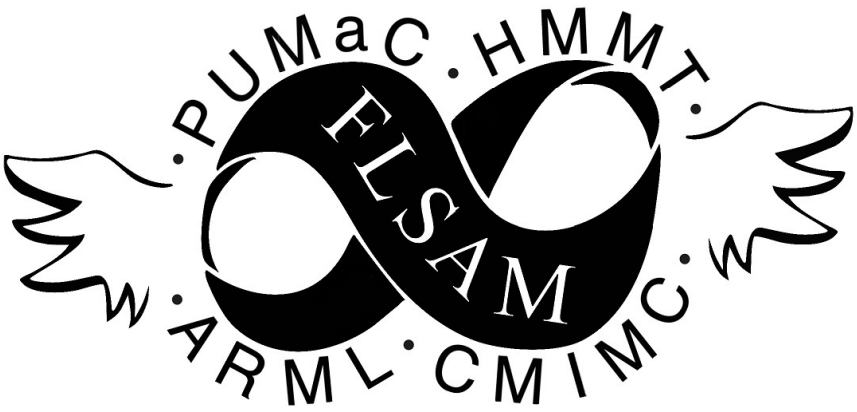
Florida Student Association of Mathematics



2024 SMT/ARML Tryout Solutions

March 2024

Round 1

Name: _____

1. 10 **2.** 5

1. There are 3 ways to permute BNB and $\binom{4}{3}$ ways to place the three $O's$ between these letters so that they're not adjacent. However, this counts two cases where the permutation of BNB has adjacent B's and there is no O in between them, so the answer is $3 \cdot \binom{4}{3} - 2 = \boxed{10}$.

2. Since both $\angle AIB$ and $\angle ADB$ are 120 degreees (the latter from Law of Cosines), we know A, I, D, B are concyclic. Thus, using Ptolemy's theorem gets $DI = \frac{2}{\sqrt{3}}$ for a final answer of $\boxed{5}$.

Round 2

Name: _____

3. 112 **4.** 57

3. Note that $CXZA$ is a parallelogram so $CX = ZA = XY = 7$ and $AX = \sqrt{(12-7)^2 + 12^2} = 13$. Let P be the projection from Y to AX; by angle chasing we can find that $\triangle XYP \sim \triangle ABX$. Then $XP = 7 \cdot \frac{5}{13} = \frac{35}{13}$ and $YZ = 13 - 2 \cdot \frac{35}{13} = \frac{99}{13}$, for an answer of $\boxed{112}$.

4. The region is a kite, with one diagonal of length 22. The two graphs intersect in the first quadrant at $x = \frac{22}{a+b}$, so the other diagonal has length $2x = \frac{44}{a+b}$ by symmetry. The area is then

$$\frac{22 \cdot 2x}{2} = 11x = \frac{22^2}{a+b} = 2024 \implies a+b = \frac{11}{46} \implies \boxed{57}.$$

Round 3

Name: _____

5. 94816 **6.** 324

5. Clearly, $A = 1$, implying that $S \geq 8$. Since we want to maximize $\overline{FLSAM}$, take $F = 9$ and $S = 8$. Then

$$\begin{aligned} 1000 + \overline{RML} = \overline{ARML} = \overline{SMT} + \overline{AFT} &= (800 + 10M + T) + (190 + T) \\ &= 990 + 10M + 2T, \end{aligned}$$

so $10 + \overline{R0L} = 2T$. Then $R = 0$, the maximum value of T is 7, and $L = 4$. There is no restriction on M, so it has maximum value 6, and the answer is $\boxed{94816}$.

6. Note $r_1 + r_2 + r_3 = -k$. WLOG, assume $r_1 + r_2 = 2r_3, 3r_3 = -k$. By the fact that $f(r_3) = 0, r_3 = -\frac{k}{3}$, we could attain $f(-\frac{k}{3}) = -\frac{k^3}{27} + \frac{k^3}{9} - \frac{20k}{3} + 24 = 0$, $\frac{2k^3}{27} - \frac{20k}{3} + 24 = 0$. Consequently, the magnitude of the product of all possible k is $|-\frac{c}{a}| = \boxed{324}$.

Round 4

Name: _____

7. 6 **8.** 8648

7. Considering $\pmod 7$, a perfect cube cannot be $9 \equiv 2 \pmod 7$. Thus, checking n from 1 to 6 gives $n = \boxed{6}$ as our only solution. (Alternatively, one can find $n \geq 9$ fails as 9 will divide the sum but not 27.)

8. The condition on the intersection point is that the following equation has one solution:

$$g(x) = (a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0.$$

If $a_1 = a_2$, then we must have $b_1 - b_2 \neq 0$. (If $b_1 = b_2$, then there would either be 0 intersection points or infinitely many.) This gives $6 \cdot (6 \cdot 5) \cdot 6^2 = 6480$ pairs of polynomials (f_1, f_2).

If $a_1 \neq a_2$, then the x-coordinate of the intersection point must be a double root of $g(x)$. Therefore, the discriminant is 0: $(b_1 - b_2)^2 = 4(a_1 - a_2)(c_1 - c_2)$. Clearly, $b_1 - b_2$ must be an even number, and we can casework on the value of $|b_1 - b_2|$. The cases are shown below, with the possible pairs $(a_1 - a_2, c_1 - c_2)$ on the RHS.

$$|b_1 - b_2| = 4 \implies \pm(4,1), \pm(2,2), \pm(1,4)$$

$$|b_1 - b_2| = 2 \implies \pm(1,1)$$

$$|b_1 - b_2| = 0 \implies (\pm n, 0) \text{ for all } n \neq 0$$

The first case, $b_1 - b_2 = \pm 4$ has 4 choices for (b_1, b_2) and $2(2 \cdot 5 + 4 \cdot 4 + 5 \cdot 2) = 72$ choices for (a_1, a_2, c_1, c_2). The second case, $b_1 - b_2 = \pm 2$ has 16 choices for (b_1, b_2) and $2(5 \cdot 5) = 50$ choices for (a_1, a_2, c_1, c_2). The last case, $b_1 - b_2 = 0$ has 6 choices for (b_1, b_2), 6 choices for (c_1, c_2), and $6 \cdot 5 = 30$ choices for (a_1, a_2).

Therefore, the answer is $6480 + 4 \cdot 72 + 16 \cdot 50 + 6 \cdot 6 \cdot 30 = \boxed{8648}$.

Round 5

Name: _____

9. 768 **10.** 100

9. Let the reflection of A over $\overline{CD}$ be E. Notice E lies on side $\overline{BC}$. Suppose N is the midpoint of $\overline{BC}$. Then, $\frac{BN}{NE} = \frac{1}{3} = \frac{BA}{AE} \implies BA = \frac{1}{3} \cdot 2 \cdot 24 = 16$. Then, $BD^2 = AB^2 + AD^2 = 16^2 + (24^2 - 8^2) = \boxed{768}$.

10. We combine the terms with equivalent denominators, so the sum becomes

$$\sum_{\text{cyc}} \frac{a+c}{b^3+b} = \sum_{\text{cyc}} \frac{-1-b}{b^3+b} = -\sum_{\text{cyc}} \frac{b+1}{b^3+b},$$

where the first equality uses Vieta's $a+b+c = -1$.
Note that, for $x = a, b, c$, we have $x^3 + x = -x^2 + x - 1 = -(x^2 - x + 1)$, making the sum equal to $\sum_{\text{cyc}} \frac{b+1}{b^2-b+1}$. Also, $x^2 - x + 1 = \frac{x^3+1}{x+1} = \frac{-x^2}{x+1}$, so

$$\sum_{\text{cyc}} \frac{b+1}{b^2-b+1} = -\sum_{\text{cyc}} \frac{(b+1)^2}{b^2} = -\sum_{\text{cyc}} 1 + \frac{2}{b} + \frac{1}{b^2}.$$

Now, we just evaluate each of these three sums. By Vieta's, $a^2b^2 + b^2c^2 + c^2a^2 = (ab+bc+ca)^2 - 2abc(a+b+c) = 0 - 2(-1)(-1) = -2$. This means $\sum_{\text{cyc}} \frac{1}{b^2} = \frac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} = \frac{-2}{(-1)^2} = -2$. Also, $\sum_{\text{cyc}} \frac{1}{b} = \frac{ab+bc+ca}{abc} = 0$. Therefore, the value of M is

$$M = -\sum_{\text{cyc}} 1 + \frac{2}{b} + \frac{1}{b^2} = -(3+0-2) = -1.$$

The answer is then $\boxed{100}$.

Round 6

Name: _____

11. 682 **12.** 41

11. Let the smallest and largest elements be m and M respectively. For $m \leq 5$, we can have $M = 11-m, 12-m, \ldots, 10$. This gives us

$$\begin{aligned}\sum_{m=1}^{5}\sum_{M=11-m}^{10} 2^{M-m-1} &= \sum_{m=1}^{5}\sum_{k=10-2m}^{9-m} 2^k \\ &= \sum_{m=1}^{5}(2^{10-m}-2^{10-2m}) \\ &= \left(\sum_{m=1}^{5} 2^{10-m}\right) - \left(\sum_{m=1}^{5} 2^{10-2m}\right) \\ &= (2^{10}-2^5) - \left(\frac{2^{10}-1}{3}\right)\end{aligned}$$

For $m > 5$, any set works, and we get $2^5 - 1$ possibilities. The total number of possibilities is $(2^{10}-2^5) - \left(\frac{2^{10}-1}{3}\right) + (2^5-1) = \boxed{682}$.

12. Since $|z_k| = 1$, we know that $\frac{1}{z_k} = \overline{z_k}$. Plugging this in gets $z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1} = 1$. Conjugating, we find

$$\frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_1}{z_3} = \frac{z_1}{z_2}\cdot\frac{z_2}{z_3} + \frac{z_2}{z_3}\cdot\frac{z_3}{z_1} + \frac{z_3}{z_1}\cdot\frac{z_1}{z_2} = 1.$$

Thus, the fractions $\frac{z_1}{z_2}$, $\frac{z_2}{z_3}$, $\frac{z_3}{z_1}$ are the roots of $x^3 - x^2 + x - 1$, which are 1, i, $-i$. WLOG, let $z_2 = z_3 = \pm i \cdot z_1$, leaving $|9z_1 + 15z_2 + 25z_3| = |z_2||40 + 9i| = \boxed{41}$.

Round 7

Name: _____

13. 93 **14.** 2004

13. Note that $BD \parallel CE$, so we have similar triangles $BDP \sim CEP$, where P is the intersection point of ℓ with BC. Note that $BP = 13$ and $CP = 15$, so if $DP = 13x$, then $PE = 15x$. By Power of a Point, $GP \cdot PE = BP \cdot PC$ while $FP \cdot PD = BP \cdot PC$, meaning $GP = \frac{13 \cdot 15}{15x} = \frac{13}{x}$ while $FP = \frac{15}{x}$.

To find x, note that $\cos A = \frac{33}{65}$, so $\cos(A/2) = \frac{7}{\sqrt{65}}$. This means $DE = (30 - 26)\cos(A/2) = \frac{28}{\sqrt{65}}$, so $DP + PE = 28x = \frac{28}{\sqrt{65}} \implies x = \frac{1}{\sqrt{65}}$.

Thus, the length $FG = \frac{28}{x} = 28\sqrt{65}$, giving the answer $\boxed{93}$.

14. We consider the permutation formed by the numbers on the board at each step. Consider the switcheroo swapping the ath and bth numbers on the board. All cycles except those containing a or b are unaffected.

Suppose a and b occur in the same cycle. Let that cycle be

$$(a, a_1, a_2, \ldots, a_i, b, b_1, b_2, \ldots, b_j)$$

After the switcheroo, this breaks into the cycles

$$(a, b_1, b_2, \ldots, b_j)(b, a_1, a_2, \ldots, a_i)$$

Suppose a and b occur in different cycles. Let those cycles be

$$(a, a_1, a_2, \ldots, a_i)(b, b_1, b_2, \ldots, b_j)$$

After the switcheroo, they join into the cycle

$$(a, b_1, b_2, \ldots, b_j, b, a_1, a_2, \ldots, a_i)$$

So, each switcheroo can increase or decrease the number of cycles by 1. In the beginning, there are 2024 cycles. By the end, all cycles have length at least 101, and so there are at most 20. Then at least $2024 - 20 = \boxed{2004}$ switcheroos have occurred.

For a valid sequence of 2004 moves, we toggle numbers in positions $(101k + 1, 101k + 2), (101k + 2, 101k + 3), \ldots, (101k + 100, 101k + 101)$ in this order for $k = 0, 1, \ldots, 19$, then $(2020, 2021), \ldots, (2023, 2024)$. This results in 19 cycles of length 101 and 1 of length 105, as desired.