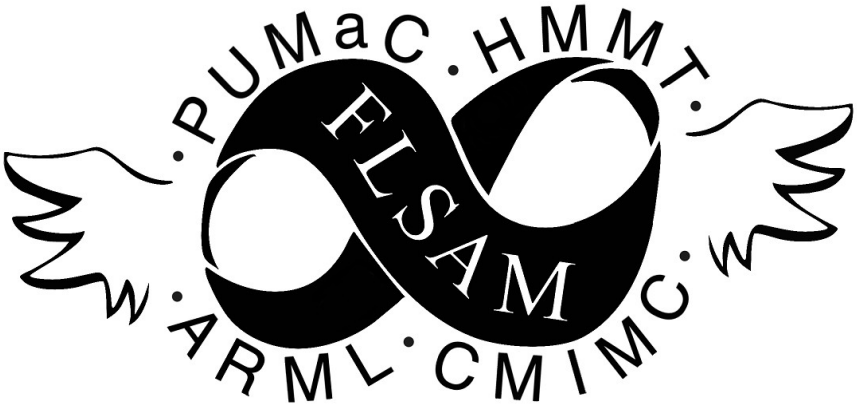
Florida Student Association of Mathematics



2023 Winter Florida Online Math Open

December 30, 2023 - January 6, 2024

Problem Contributors

Test Coordinators

- Alex Li (`@idk5799`)
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Curriculum Committee

- Sailalitha Kodukula
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Thank you to everyone who submitted problems!

Rules

1. This is a 30 problem, team-based competition that takes place over one week. Answers must be submitted before 7PM EDT on January 6th.
2. This is a team event; collaboration with your teammates is encouraged. Discussion about the contest or its problems is not allowed between members of different teams or anyone not competing.
3. Each correct answer is worth 1 point. Incorrect or omitted answers are both worth 0 points.
4. The difficulty of each question will be determined by how many teams correctly answer it. Questions with fewer solves will be considered more difficult. Ties will be broken by considering the most difficult question each team correctly answers. The team that correctly answered the more difficult question wins the tiebreaker. If necessary, the process will move to the second most difficult question each team correctly answers and so on.
5. You may use four function calculators. All problems on this test can be solved without a calculator. Drawing aids such as graph paper, rulers, compasses, and protractors are permitted, but electronic programs such as graphing calculators, Desmos, or Geogebra are not.
6. **All answers are nonnegative integers**. Any non-integer quantities will be converted to integers by instructions in the problem. Fractions of the form $\frac{m}{n}$ and radicals of the form $p\sqrt{q}$ are assumed to be in simplest form (that is, $\gcd(m,n) = 1$, q is squarefree, and all variables are positive integers). Isolated radicals of the form $\sqrt{k}$ do not necessarily follow this rule.
7. Make sure to check both the `#flomo-updates` channel in Discord and emails from `flsamflorida@gmail.com` for any clarifications or updates. If you have any questions, please contact the testing coordinators at `flsamflorida@gmail.com` or through Discord DMs to the given tags.

In addition, the following approximations and definitions may be useful:

The floor function, $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

The ceiling function, $\lceil x \rceil$, is the least integer greater than or equal to x.

$$\pi \approx 3.14159265359$$

$$e \approx 2.71828182846$$

Problems

1. Given that x is a real number, find the minimum value of $20|x+20|+23|x+23|+24|x+24|$.
2. Reals F, L, S, A, M form a geometric sequence satisfying $FL = 36$ and $SAM = 64$. What is $SALSAS$?
3. Jiggy and Ishika are playing a game. Jiggy goes first, and they alternate rolling a standard die until one of them rolls a 6. Whoever rolled the 6 then flips a fair coin, and they alternate flipping the coin until one of them flips heads. The person who flips heads is the winner. If the probability that Jiggy wins is $\frac{m}{n}$ in simplest form, what is $100m+n$?
4. In rectangle $ABCD$ has $AB = 30$, M is the midpoint of $\overline{BC}$, and N is the midpoint of $\overline{AM}$. If $\triangle NCD$ is equilateral, what is BC^2?
5. How many distinct ordered pairs of polynomials $(f(x), g(x))$ with integer coefficients satisfy
$$f(x)g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7?$$
6. Suppose convex quadrilateral $JIGY$ satisfies $JI = JY = 50$, $\angle J = 72^\circ$, and $\angle G = 144^\circ$. Find JG^2.
7. 100 consecutive positive integers are written on a blackboard. Andrew erases one of the numbers and replaces it with another positive integer. Afterwards, if the median of the numbers on the board is 69, what is the sum of all possible values of the smallest number on the board?
8. Find the unique 4-digit positive integer $\overline{ABCD}$, where A, B, C, D are digits, that satisfies
$$\overline{ABCD} = 2A + \log_2 B + 2C + D^B.$$
9. Darsh is on the first floor of a 10-story building. He flips a coin: if it's heads, he goes up one floor on the elevator; if it's tails, he goes up two floors on the elevator. He repeats this until he's on the 10th floor, where his best friend Tiger is (if Darsh is on the 9th floor and flips tails, he also only goes up one floor). However, Tiger gets annoyed if Darsh visits Ishika on the 9th floor on his way up. If the probability that Tiger gets annoyed is $\frac{m}{n}$ in simplest form, what is $100m+n$?
10. A right triangle has side lengths 3, 4, and 5. Aaron draws the incircle and circumcircle of this triangle. If the distance from the circumcenter to the center of dilation that takes the incircle to the circumcircle is d and $d^2 = \frac{m}{n}$ for relatively prime positive integers m and n, find $100m+n$.
11. A hexagon with alternating side lengths of 20 and 23 is inscribed in a circle. If the area of the circle can be expressed as $k\pi$ where k is an integer, find k.
12. Alex and Aaron each throw 2 darts at a bullseye. Each dart lands at a a random point on the target and Alex's first dart lands closer to the bullseye than Aaron's first dart. If the probability that Alex's closest shot to the bullseye is closer than Aaron's best is $\frac{m}{n}$ in simplest form, compute $100m+n$.
13. Compute the sum
$$\sum_{0 \le i < j \le 100} \left\lfloor \frac{10.5+i}{j} \right\rfloor.$$
14. Given triangle ABC with $AB = 5$, $BC = 6$, $CA = 7$, consider the unique point P that minimizes $3AP + 2BP + CP$. The circumradius R of PBC satisfies $R^2 = \frac{m}{n}$ when written in simplest form. Compute $100m+n$.
15. If S is a set containing 2024 elements, then
$$\sum_{A \subseteq S} \sum_{B \subseteq S} |A \cap B|$$
has n positive integer divisors. What is n?

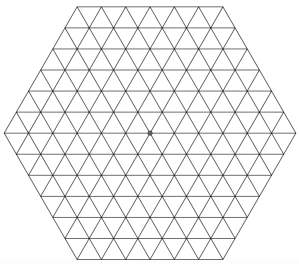
16. Consider acute triangle ABC. Let M denote the midpoint of $\overline{AB}$, and let O be the point on $\overline{CM}$ such that $\angle AOB = 120^\circ$. Given $\angle BAC = 60^\circ$, $AO = 5$, and $BO = 8$, the length of CM can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

17. Let (a, b) be nonzero numbers satisfying $a^2 + b^2 = 6$ and $\frac{a^3}{b} + 6ab + \frac{b^3}{a} = 40$. If the sum of all the possible values of $|a + b|$ can be expressed as $\sqrt{p} + \sqrt{q}$ for integers $p > q$, find $100p + q$.

18. Let $ABCD$ be a rhombus with area $\mathcal{A}$, side length 4, and $\angle A < 90^\circ$. Suppose that the line through A perpendicular to BC and the line through D perpendicular to AB intersect on line BC. If $\mathcal{A}^2$ can be expressed as $a\sqrt{b} - c$ for positive integers a, b, c with b square-free, what is $100a + 10b + c$?

19. Five lines split the plane into a number of regions. Find the maximum number of these regions, bounded or unbounded, that one can color so that no two colored regions share an edge.

20. A spider is initially at the center of its hexagonal web of side length 6. The web is made of equilateral triangles of side length 1, as shown below. Each minute, the spider moves 1 unit along an adjacent edge. Suppose the probability that the spider reaches the edge of the web after six minutes is $\frac{m}{n}$ when written in simplest form. Find $100m + n$.



21. Triangle ABC has side lengths $AB = 30$, $BC = 70$, $CA = 68$ and incenter I. The incircle touches BC at D, CA at E, and AB at F. Let ℓ_1 be the line through D perpendicular to EF, and let ℓ_2 be the line through A perpendicular to BC. If $\ell_1 \cap \ell_2 = P$ and $PI \cap BC = Q$, find IQ^2.

22. Find $r_1^5 + r_2^5 + \cdots + r_5^5$ where r_1, r_2, r_3, r_4, and r_5 are the distinct roots of

$$(x - \sqrt[5]{2021})(x - \sqrt[5]{2022})(x - \sqrt[5]{2023})(x - \sqrt[5]{2024})(x - \sqrt[5]{2025}) = 7.$$

23. For positive integers $m \leq n$, let $f(m, n)$ denote the number of integer divisors d of $mn + 1$ with $m < d < n$. Compute $f(0, 2023) + f(1, 2023) + \cdots + f(2023, 2023)$.

24. Find the sum of xyz over all ordered triples (x, y, z) of positive integers satisfying

$$xy + yz + zx - 15\sqrt{x^2 + y^2 + z^2} = 69.$$

25. For some complex numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ satisfying

$$23\alpha_1 - 20\alpha_2 - 3\alpha_3 = -23\beta_1, \quad 5\alpha_1 - 6\alpha_2 + \alpha_3 = 6\beta_2,$$

there exist unique real numbers r, s_1, s_2, s_3 such that

$$r\alpha_1 + (1 - r)\alpha_2 - \alpha_3 = \beta_3, \quad \alpha_1 + s_1\beta_1 = \alpha_2 + s_2\beta_2 = \alpha_3 + s_3\beta_3.$$

If $r + s_1 + s_2 + s_3$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $100m + n$.

26. A permutation of the first 2023 integers is chosen at random. If the expected value of the square of the number of inversions can be expressed as $\frac{m}{n}$ in simplest terms, find the sum of the distinct prime divisors of m. (Note: an *inversion* is a move that swaps the positions of two numbers in a permutation.)

27. Let S be the sum of

$$\binom{a_1}{1000} + \binom{a_2}{1000} + \cdots + \binom{a_{10}}{1000}$$

over all tuples $(a_1, a_2, \ldots, a_{10})$ of nonnegative integers such that $a_1 + a_2 + \cdots + a_{10} = 2023$. Find the largest nonnegative integer k for which 2^k divides S.

28. Consider reals $a_1, a_2, \ldots, a_5 \geq 0$ with a sum of 1. Define P and Q as follows:

$$P = \min\{a_1, a_2\} + 2 \cdot \min\{a_2, a_3\} + \cdots + 5 \cdot \min\{a_5, a_1\}$$
$$Q = \max\{a_1, a_2\} + 2 \cdot \max\{a_2, a_3\} + \cdots + 5 \cdot \max\{a_5, a_1\}$$

When P reaches its maximum, the range of all possible values of Q is $[a, b]$. Find $\lfloor 100(a+b) \rfloor$.

29. Find the number of distinct values that

$$\frac{i^4 - 1}{i^2}$$

takes modulo 2023 over all i such that $\gcd(i, 2023) = 1$.

30. Consider cyclic quadrilateral $ABCD$ with integer side lengths. Let $F = AC \cap BD$, and denote the circumcenters of $CBF, ABCD, DCF$ as H, I, J, respectively. Given that $AB = AD = 6$, $AF = 3$, and $CF = 9$, the area of $\triangle HIJ$ can be expressed in simplest form as $\frac{p\sqrt{q}}{r}$. Compute $100p + 10q + r$.