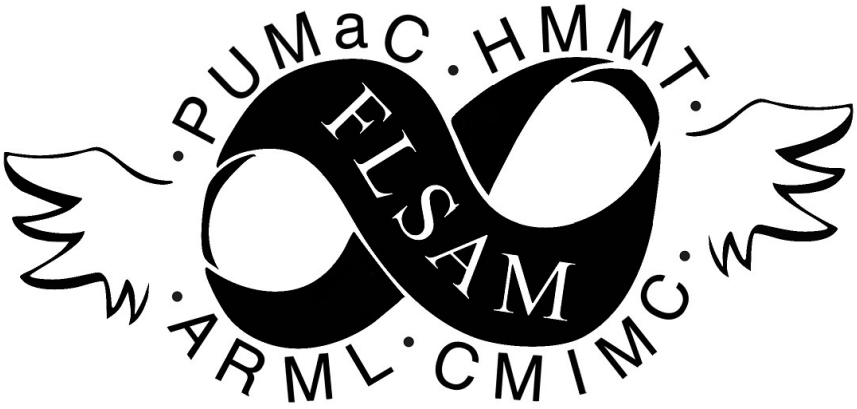
# Florida Student Association of Mathematics




## 2023 Winter Florida Online Math Open Solutions

December 30, 2023 - January 6, 2024

**1.** Given that $x$ is a real number, find the minimum value of $20|x+20|+23|x+23|+24|x+24|$.

**Answer:** 84

*Proposed by Aaron Hu*

Clearly, $-24 \le x \le -20$, so the expression equals

$$-20(x+20)+23|x+23|+24(x+24) = 23|x+23|+4x+176 = 23|x+23|+4(x+23)+84.$$

Then the minimum is $\boxed{84}$, attained when $x=-23$.

**2.** Reals $F, L, S, A, M$ form a geometric sequence satisfying $FL = 36$ and $SAM = 64$. What is $SALSAS$?

**Answer:** 9216

*Proposed by Aaron Hu*

Let $F = a$, $L = ar$, and so on until $M = ar^4$. Then

$$36 = FL = a^2r, \quad 64 = SAM = a^3r^9 \implies ar^3 = 4,$$

so

$$SALSAS = a^6r^{13} = (a^2r)(ar^3)^4 = 36 \cdot 4^4 = \boxed{9216},$$

as desired.

**3.** Jiggy and Ishika are playing a game. Jiggy goes first, and they alternate rolling a standard die until one of them rolls a 6. Whoever rolled the 6 then flips a fair coin, and they alternate flipping the coin until one of them flips heads. The person who flips heads is the winner. If the probability that Jiggy wins is $\frac{m}{n}$ in simplest form, what is $100m+n$?

**Answer:** 1733

*Proposed by Sailalitha Kodukula*

First, the probability that Jiggy is the first to roll a 6 on the dice is $\frac{6}{11}$, as $p = \frac{1}{6} + \frac{5}{6}(1-p)$. Then, the probability that the first to flip a coin is the first to flip heads satisfies $q = \frac{1}{2} + \frac{1}{2}(1-q)$, so $q = \frac{2}{3}$.

Then, the probability Jiggy wins is $\frac{6}{11} \cdot \frac{2}{3} + \frac{5}{11} \cdot \frac{1}{3} = \frac{17}{33}$, for an answer of $\boxed{1733}$.

**4.** In rectangle $ABCD$ has $AB = 30$, $M$ is the midpoint of $\overline{BC}$, and $N$ is the midpoint of $\overline{AM}$. If $\triangle NCD$ is equilateral, what is $BC^2$?

**Answer:** 1200

*Proposed by Aaron Hu*

Let $P$ denote the midpoint of $\overline{CD}$. Then

$$15\sqrt{3} = \frac{\sqrt{3}}{2} \cdot CD = NP = \frac{1}{2}(AD+MC) = \frac{3}{4} \cdot BC \implies BC = 20\sqrt{3},$$

and the answer is $\boxed{1200}$.

**5.** How many distinct ordered pairs of polynomials $(f(x), g(x))$ with integer coefficients satisfy

$$f(x)g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7?$$

**Answer:** 16

*Proposed by Sailalitha Kodukula*

One can factor $f(x)g(x) = (x+1)(x^2+1)(x^4+1)$, so there is one possible polynomial for $f(x)$ with any degree from 0 to 7. Also, the sign of $f(x)$ may be positive or negative. Thus, there are $8 \cdot 2 = \boxed{16}$ total ordered pairs.

**6.** Suppose convex quadrilateral $JIGY$ satisfies $JI = JY = 50$, $\angle J = 72^\circ$, and $\angle G = 144^\circ$. Find $JG^2$.

**Answer:** 2500

*Proposed by Jayden Lang*

Since $360 - 72 = 2 \cdot 144$, a circle centered on $J$ passes through $I, G$, and $Y$. Thus, $JG = JI = JY = 50 \implies JG^2 = \boxed{2500}$.

**7.** 100 consecutive positive integers are written on a blackboard. Andrew erases one of the numbers and replaces it with another positive integer. Afterwards, if the median of the numbers on the board is 69, what is the sum of all possible values of the smallest number on the board?

**Answer:** 210

*Proposed by Aaron Hu*

There are four cases where the new median is 69:

- The original set of numbers was $20, \ldots, 119$, and one of $70, \ldots, 119$ was replaced with 69.
- The original set of numbers was $19, \ldots, 118$, and one of $19, \ldots, 68$ was replaced with 69.
- The original set of numbers was $20, \ldots, 119$, and 69 is replaced with a number less than 69.
- The original set of numbers was $19, \ldots, 118$, and 69 is replaced with a number greater than 69.

Then the smallest number on the board after replacement can be any positive integer less than or equal to 20, and the answer is $1 + 2 + \cdots + 20 = \boxed{210}$.

**8.** Find the unique 4-digit positive integer $\overline{ABCD}$, where $A, B, C, D$ are digits, that satisfies

$$\overline{ABCD} = 2A + \log_2 B + 2C + D^B.$$

**Answer:** 2407

*Proposed by Yuhan Niu*

$B$ must be $1, 2, 4, 8$. Since the majority of the size of the number comes from $D^B$, $B$ is not 1 or 2. $B$ also cannot be 8. If $B$ is 8, we can simplify $2A + \log_2 B + 2C$ to be odd. From here, if D is odd, the LHS is odd, but the RHS is odd+odd which is even, so D cannot be odd. If D is even, the LHS is even, but the RHS is odd+even, so D cannot be any number. Therefore, $B$ cannot be 8.

After substituting $B = 4$, we can create bounds of $998 \leq 2A + 2C + D^4 \leq 9997$. Since $A$ can be 1-9 and $C$ can be 0-9, we can say $996 \leq D^4 \leq 9961$. Only possible values of $D$ are $6, 7, 8$ and 9. Taking each of them to the 4th power, only for $D = 7$ is it possible that $B = 4$. With more manipulation, we get that $\overline{ABCD} = \boxed{2407}$.

**9.** Darsh is on the first floor of a 10-story building. He flips a coin: if it's heads, he goes up one floor on the elevator; if it's tails, he goes up two floors on the elevator. He repeats this until he's on the 10th floor, where his best friend Tiger is (if Darsh is on the 9th floor and flips tails, he also only goes up one floor). However, Tiger gets annoyed if Darsh visits Ishika on the 9th floor on his way up. If the probability that Tiger gets annoyed is $\frac{m}{n}$ in simplest form, what is $100m + n$?

**Answer:** 17356

*Proposed by Sailalitha Kodukula*

We use recursion. Let $p_i$ be the probability if Darsh starts on the $i^{\text{th}}$ floor. Note $p_9 = 1$ and $p_8 = \frac{1}{2}$. For $1 \leq i \leq 7$, we find $p_i = \frac{1}{2}(p_{i+1} + p_{i+2})$.

So, working backwards, $p_7 = \frac{3}{4} = \frac{2}{3} + \frac{1}{12}$, $p_6 = \frac{5}{8} = \frac{2}{3} - \frac{1}{24}$ and $p_i = \frac{2}{3} + (-1)^{(i+1)} \frac{1}{3 \cdot 2^{(9-i)}}$.

The desired answer is $p_1 = \frac{2}{3} + \frac{1}{3 \cdot 2^8} = \frac{171}{256}$, giving $17100 + 256 = \boxed{17356}$.

**10.** A right triangle has side lengths 3, 4, and 5. Aaron draws the incircle and circumcircle of this triangle. If the distance from the circumcenter to the center of dilation that takes the incircle to the circumcircle is $d$ and $d^2 = \frac{m}{n}$ for relatively prime positive integers $m$ and $n$, find $100m + n$.

**Answer:** 12536

*Proposed by Alex Li*

The inradius is 1. Drop perpendiculars from the incenter $I$ and circumcenter $O$ to the side of length 4. The feet of the perpendiculars are distance 1 apart. Also, the difference in lengths of the perpendiculars is $1/2$, so the distance $OI = \sqrt{5/4}$.

To find the center of homothety (dilation) $D$, note that the ratio of the radii is $1 : 2.5 = 2 : 5$. Then, the ratio $DI : DO = 2 : 5$ as well. Note that $D$ should be on line $OI$ but outside line segment $OI$. Thus, $OI = \sqrt{5/4}$ implies $DI = \frac{2}{3} \cdot OI = \sqrt{5}/3$, and $DO = \frac{5\sqrt{5}}{6}$.

The answer is then $12500 + 36 = \boxed{12536}$.

**11.** A hexagon with alternating side lengths of 20 and 23 is inscribed in a circle. If the area of the circle can be expressed as $k\pi$ where $k$ is an integer, find $k$.

**Answer:** 463

*Proposed by Aaron Zhang*

The hexagon is clearly equiangular. We can apply law of cosines to find the side length, $s$, of an equilateral triangle with alternating vertices of the hexagon. $s = \sqrt{20^2 + 23^2 - 2 \cdot 20 \cdot 23 \cdot \cos 120^\circ} = \sqrt{1389}$. Then, the radius of the circle is $r = \sqrt{1389} \cdot \frac{\sqrt{3}}{3} = \sqrt{463}$, and $k = r^2 = \boxed{463}$.

**12.** Alex and Aaron each throw 2 darts at a bullseye. Each dart lands at a a random point on the target and Alex's first dart lands closer to the bullseye than Aaron's first dart. If the probability that Alex's closest shot to the bullseye is closer than Aaron's best is $\frac{m}{n}$ in simplest form, compute $100m + n$.

**Answer:** 304

*Proposed by Jessica Wan*

Denote Alex as A and Aaron as B. Note that we can fix the four distances to the target and consider only the ordering of the distances. There are six possible orderings (closest to furthest) and up to 4 ways to designate the first and second darts for A and B: AABB (4 ways), ABAB (3 ways), ABBA (2 ways), BAAB (2 ways), BABA (1 way), BBAA (0 ways).

Then, the conditional probability of A's best shot beating B's is $\frac{4+3+2}{4+3+2+2+1+0} = \frac{3}{4}$, giving an answer of $\boxed{304}$.

**13.** Compute the sum

$$\sum_{0 \leq i < j \leq 100} \left\lfloor \frac{10.5 + i}{j} \right\rfloor.$$

**Answer:** 1000

*Proposed by Kevin Liu*

We have that

$$\sum_{0\le i<j\le 100}\left\lfloor\frac{10.5+i}{j}\right\rfloor=\sum_{j=1}^{100}\sum_{i=0}^{j-1}\left\lfloor\frac{10.5}{j}+\frac{i}{j}\right\rfloor=\sum_{j=1}^{100}\left\lfloor j*\frac{10.5}{j}\right\rfloor=\sum_{j=1}^{100}10=10*100=\boxed{1000}.$$

**14.** Given triangle $ABC$ with $AB=5$, $BC=6$, $CA=7$, consider the unique point $P$ that minimizes $3AP+2BP+CP$. The circumradius $R$ of $PBC$ satisfies $R^2=\frac{m}{n}$ when written in simplest form. Compute $100m+n$.

**Answer:** 122596

*Proposed by Aaryan Vaishya*

The triangle inequality gives

$$2AP+2BP\ge 2AB,\quad AP+CP\ge AC\implies 3AP+2BP+CP\ge 2AB+AC=17,$$

with equality only when $P$ lies on segment $AB$ and $AC$ respectively. Thus, $P=A$ and we find the circumradius of triangle $ABC$ equals $\frac{abc}{4[ABC]}=\frac{5\cdot 6\cdot 7}{4\cdot 6\sqrt{6}}=\frac{35}{4\sqrt{6}}$. Then, $\frac{m}{n}=\frac{1225}{96}$ so our answer is $\boxed{122596}$.

**15.** If $S$ is a set containing 2024 elements, then

$$\sum_{A\subseteq S}\sum_{B\subseteq S}|A\cap B|$$

has $n$ positive integer divisors. What is $n$?

**Answer:** 16200

*Proposed by Kevin Liu*

WLOG, let $S$ be the set of the first 2024 positive integers. Then

$$\sum_{A\subseteq S}\sum_{B\subseteq S}|A\cap B|=\sum_{A\subseteq S}\sum_{B\subseteq S}\sum_{i\in A\cap B}1=\sum_{i=1}^{2024}\sum_{i\in A\subseteq S}\sum_{i\in B\subseteq S}1.$$

That is equal to $2024\cdot 2^{2023}\cdot 2^{2023}=2^{4049}\cdot 11\cdot 23$, which has $4050\cdot 2\cdot 2=\boxed{16200}$ divisors.

**16.** Consider acute triangle $ABC$. Let $M$ denote the midpoint of $\overline{AB}$, and let $O$ be the point on $\overline{CM}$ such that $\angle AOB=120^\circ$. Given $\angle BAC=60^\circ$, $AO=5$, and $BO=8$, the length of $CM$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers $m,n$. Find $100m+n$.

**Answer:** 90458

*Proposed by Tom Zhang*

Reflect $O$ about $M$ to $O'$, it is clear that $AO'BO$ is a parallelogram, as a result, $\angle O'AO=60^\circ, AO'=8, BO'=5$. By the law of cosines, we attain $OO'=7$. Let $AC=b, CO=c$, by Stewart on $\triangle AO'C$ with cevian $AO$, we have $64c+7b^2=(7+c)(7c+25)$. Note that since $\angle O'AM=\angle OAC$, we can conclude that $AM, AO$ are isogonal wrt the angle bisector of $\angle O'AC$. Thus, $(\frac{8}{b})^2=\frac{\frac{7}{2}}{(c+\frac{7}{2})}\cdot\frac{7}{c}$. Combine two equations together, we have $c=\frac{175}{79}$, $CM=c+OM=\frac{175}{79}+\frac{7}{2}=\frac{903}{158}\implies\boxed{90458}$.

**17.** Let $(a,b)$ be nonzero numbers satisfying $a^2+b^2=6$ and $\frac{a^3}{b}+6ab+\frac{b^3}{a}=40$. If the sum of all the possible values of $|a+b|$ can be expressed as $\sqrt{p}+\sqrt{q}$ for integers $p>q$, find $100p+q$.

**Answer:** 2408

*Proposed by Jayden Lang*

Notice that $\frac{a^3}{b} + 6ab + \frac{b^3}{a} = \frac{(a^2+b^2)^2}{ab} + 4ab = \frac{36}{ab} + 4ab = 40$. This simplifies to $(ab)^2 - 10ab + 9 = 0$, which factors into $(ab-9)(ab-1) = 0$. If $ab = 9$, $a^2 + 2ab + b^2 = 24 \implies |a+b| = \sqrt{24}$. If $ab = 1$, $a^2 + 2ab + b^2 = 8 \implies |a+b| = \sqrt{8}$. Thus, the answer is $2400 + 8 = \boxed{2408}$.

**18.** Let $ABCD$ be a rhombus with area $\mathcal{A}$, side length 4, and $\angle A < 90^\circ$. Suppose that the line through $A$ perpendicular to $BC$ and the line through $D$ perpendicular to $AB$ intersect on line $BC$. If $\mathcal{A}^2$ can be expressed as $a\sqrt{b} - c$ for positive integers $a, b, c$ with $b$ square-free, what is $100a + 10b + c$?

**Answer:** 12978

*Proposed by Aaron Hu*

Let $P$ denote the concurrency point, and $Q = \overline{AB} \cap \overline{DP}$. Let $x = AP = DQ$ denote the distance between pairs of opposite sides of the rhombus, and let $y = AB = AD$ denote the side length of the rhombus. Since $\triangle DAQ \sim \triangle DPA$,

$$x \cdot DP = DQ \cdot DP = AD^2 = y^2.$$

Since $\triangle DPA$ is right,

$$\sqrt{x^2 + y^2} = \sqrt{AP^2 + AD^2} = DP = \frac{y^2}{x}.$$

Dividing both sides by $x$ gives $\sqrt{1+c} = c$, where $c = \frac{y^2}{x^2}$. Then

$$\frac{16}{x^2} = \frac{y^2}{x^2} = \frac{1+\sqrt{5}}{2} = \frac{2}{\sqrt{5}-1} \implies \mathcal{A}^2 = x^2y^2 = 16 \cdot 8(\sqrt{5}-1) = 128\sqrt{5} - 128,$$

so the answer is $\boxed{12978}$.

**19.** Five lines split the plane into a number of regions. Find the maximum number of these regions, bounded or unbounded, that one can color so that no two colored regions share an edge.

**Answer:** 10

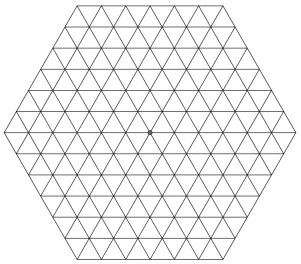
*Proposed by Jessica Wan*

Consider the five lines arranged as the diagonals of a regular pentagon. Then, color the 5 enclosed triangular regions and unbounded sectors formed by two lines. This gives $\boxed{10}$ regions.

Now, we prove 10 is maximal. First, note that there are at most 10 unbounded regions, as each line has two infinite rays and each unbounded region must be bounded by at least two. (In fact, there are exactly two). We can color at most half of these.

Next, each line has 3 finite segments, and each segment can be part of the border of at most one colored region. As each bounded region requires at least 3 segments in its boundary, we have at most $5 \cdot 3/3 = 5$ colored bounded regions. Altogether, we can color at most 10 regions, as desired.

**20.** A spider is initially at the center of its hexagonal web of side length 6. The web is made of equilateral triangles of side length 1, as shown below. Each minute, the spider moves 1 unit along an adjacent edge. Suppose the probability that the spider reaches the edge of the web after six minutes is $\frac{m}{n}$ when written in simplest form. Find $100m + n$.



**Answer:** 1564

*Proposed by Yuhan Niu*

Each move, the spider needs to move outward. If the spider is along one of the diameters of the hexagon, it has 3 moves outward, while everywhere else it only has 2. From this, we can do casework for when the spider steps of the diameter, resulting in probabilities of $\frac{3}{6^5}$, $\frac{4}{6^5}$, $\frac{8}{6^5}$, $\frac{16}{6^5}$, and $\frac{32}{6^5}$. The sum of these is $\frac{7}{864}$, so $100m + n = \boxed{1564}$.

**21.** Triangle $ABC$ has side lengths $AB = 30$, $BC = 70$, $CA = 68$ and incenter $I$. The incircle touches $BC$ at $D$, $CA$ at $E$, and $AB$ at $F$. Let $\ell_1$ be the line through $D$ perpendicular to $EF$, and let $\ell_2$ be the line through $A$ perpendicular to $BC$. If $\ell_1 \cap \ell_2 = P$ and $PI \cap BC = Q$, find $IQ^2$.

**Answer:** 505

*Proposed by Andrew Xing*

The key observation is that $APDI$ is a parallelogram. Clearly, $\ell_2$ is parallel to $ID$ because both are perpendicular to $BC$. Additionally, $AI$ is parallel to $PD$ because both are perpendicular to $EF$. Thus, $APDI$ is a parallelogram. Call $T$ the point on the incircle diametrically opposite to $D$. We also know that $ATPI$ is a parallelogram, as $TD$ passes through $I$ and $TI = ID$. Extend $AT$ until the point $S$, the $A$-extouch point, using a homothety from the incircle to the excircle.

We know that $SQ$ is equivalent to $DQ$ based on similar triangles. Furthermore, since $D$ and $S$ are the intouch and extouch points, respectively, $BD = SC$. Thus, $Q$ is the midpoint of $BC$, and we can find the length of $IQ$ by first finding the radius and using equal tangents to find $DQ$. Finding the inradius as 12 (using area formulae) and the length $DQ = 19$, we find our answer to be $\boxed{505}$.

**22.** Find $r_1^5 + r_2^5 + \cdots + r_5^5$ where $r_i$ are the roots of

$$(x - \sqrt[5]{2021})(x - \sqrt[5]{2022})(x - \sqrt[5]{2023})(x - \sqrt[5]{2024})(x - \sqrt[5]{2025}) = 7.$$

**Answer:** 10150

*Proposed by Jessica Wan*

Denote the $k^{\text{th}}$ symmetric sum of $\{r_i\}$ by $\sigma_k$ and the power sum by $p_k$. We have $\sigma_0 = 1$, $\sigma_1 = r_1 + r_2 + r_3 + r_4 + r_5$, $\ldots$, $\sigma_5 = r_1r_2r_3r_4r_5$ and $p_0 = 5$, $p_1 = \sigma_1$, and $p_k = \Sigma_{i=1}^5 r_i^k$.

Similarly, let $a_i = 2020 + i$. Let $e_k$ denote the $k^{\text{th}}$ symmetric sum of $a_i$ and $q_k$ the power sum. Now, Vieta's tells us that $\sigma_k = e_k$ for $0 \leq k \leq 4$, and $\sigma_5 = e_5 + 7$. Thus, their power sums are equal up to the fourth power as well.

Now, Newton's identities tell us that $5r_1r_2r_3r_4r_5 = \sum_{i=1}^5 (-1)^{i-1}\sigma_{5-i}p_i$. So, $p_5 = \sigma_1 p_4 - \cdots - \sigma_4 p_1 + 5r_1r_2r_3r_4r_5$. As we can write a similar equation for $\{a_i\}$, we find $p_5 = q_5 + 5(\sigma_5 - e_5) = a_5 + 35 = 5 \cdot 2023 + 35 = \boxed{10150}$.

**23.** For positive integers $m \leq n$, let $f(m,n)$ denote the number of divisors $d$ of $mn+1$ with $m < d < n$. Compute $f(0,2023) + f(1,2023) + \cdots + f(2023,2023)$.

**Answer:** 1632

*Proposed by Darsh Patel*

Consider any $m \geq 0$. Take any divisor $d$ of $2023m+1$ that is counted by $f(m,2023)$. Now, if we consider $\frac{2023m+1}{d}$, we see that

$$d > m \iff \frac{2023m+1}{d} < 2023$$

and

$$d < 2023 \iff \frac{2023m+1}{d} > m$$

Thus, we can also interpret $f(m,2023)$ as counting the number of pairs $a,b$ such that $ab = 2023m+1$ and $0 < a,b < 2023$.

Now considering the entire sum, since $ab$ cannot exceed $2023^2+1$, the sum counts the number of pairs $a,b$ with $ab \equiv 1 \pmod{2023}$ and $0 < a,b < 2023$. The number of solutions to $ab \equiv 1 \pmod{2023}$ is just the number of invertible residues modulo 2023, or $\phi(2023) = \boxed{1632}$.

**24.** Find the sum of $xyz$ over all ordered triples $(x,y,z)$ of positive integers satisfying

$$xy + yz + zx - 15\sqrt{x^2+y^2+z^2} = 69.$$

**Answer:** 6912

*Proposed by Andrew Xing*

Removing the square root, we find $(xy+yz+zx-69)^2 = 225(x^2+y^2+z^2)$. To complete the square on the right side, we add $450(xy+yz+zx)$ to both sides, yielding $(xy+yz+zx+156)^2 - 19575 = 225(x+y+z)^2$. Using difference of squares, we get

$$(xy+yz+zx+156-15(x+y+z))(xy+yz+zx+156+15(x+y+z)) = 15^2 \cdot 3 \cdot 29.$$

Since both factors are the same $0 \pmod{15}$ and the first is less than the second, we have two cases:

- The factors are 45 and 435. In this case, we can find that $x+y+z = 13$, $xy+yz+zx = 84$, so
$$x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = 13^2 - 2 \cdot 84 = 1,$$
and there are clearly no solutions here.
- The factors are 15 and 1305. In this case, we can find that $x+y+z = 43$, $xy+yz+zx = 504$. Then
$$xy = 504 - z(x+y) = 504 - (x+y)(43-(x+y)) = k^2 - 43k + 504,$$
where $k = x+y$. The quadratic with roots $x,y$ has roots $k^2 - 4(k^2-43k+504) = -3k^2+172k-2016$. From here, we need only test $30 \leq k < 43$ (by assuming $x \geq y \geq z$) to see if it yields a perfect square. The only such $k$ is 40, which gives $(x,y) = (24,16)$ and $z = 3$. Thus, $(x,y,z) = (24,16,3)$ and permutations are all solutions.

The answer is $3! \cdot 24 \cdot 16 \cdot 3 = \boxed{6912}$.

**25.** For some complex numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ satisfying

$$23\alpha_1 - 20\alpha_2 - 3\alpha_3 = -23\beta_1, \quad 5\alpha_1 - 6\alpha_2 + \alpha_3 = 6\beta_2,$$

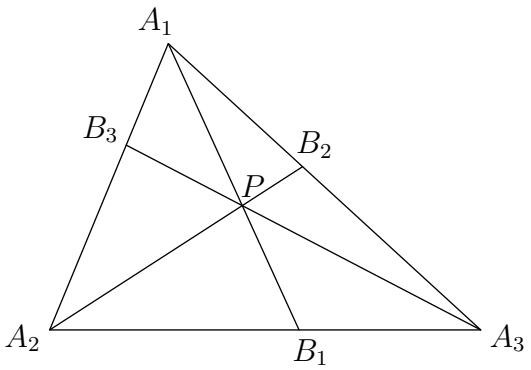
there exist unique real numbers $r, s_1, s_2, s_3$ such that

$$r\alpha_1 + (1-r)\alpha_2 - \alpha_3 = \beta_3, \quad \alpha_1 + s_1\beta_1 = \alpha_2 + s_2\beta_2 = \alpha_3 + s_3\beta_3.$$

If $r + s_1 + s_2 + s_3$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers $m, n$, compute $100m + n$.

**Answer:** 1707

*Proposed by Darsh Patel*




Let $A_1, A_2, A_3$ be the points corresponding to $\alpha_1, \alpha_2, \alpha_3$, respectively, in the complex plane. Let $B_1, B_2, B_3$ be the points corresponding to $\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3$, respectively, in the complex plane.

From the first three equations, we have that $B_1$ lies on $A_2A_3$ and similarly for $B_2$ and $B_3$. Now, $\beta_1$ is the vector $\overrightarrow{A_1B_1}$ and similarly for $\beta_2, \beta_3$. The third condition then implies that $A_1B_1, A_2B_2, A_3B_3$ concur. Let $P$ be the point of concurrence.

By Ceva's Theorem, we have

$$\frac{A_3B_1}{B_1A_2} \cdot \frac{A_1B_2}{B_2A_3} \cdot \frac{A_2B_3}{B_3A_1} = 1 \implies \frac{20}{3} \cdot \frac{1}{5} \cdot \frac{r}{1-r} = 1 \implies r = \frac{3}{7}$$

Also,

$$s_1 + s_2 + s_3 = 3 - \frac{[A_2PA_3]}{[A_2A_1A_3]} - \frac{[A_1PA_3]}{[A_1A_2A_3]} - \frac{[A_1PA_2]}{[A_1A_3A_2]} = 2$$

The desired sum is $\frac{17}{7}$, and the answer is $\boxed{1707}$.

**26.** A permutation of the first 2023 integers is chosen at random. If the expected value of the square of the number of inversions can be expressed as $\frac{m}{n}$ in simplest terms, find the sum of the distinct prime divisors of $m$. (Note: an *inversion* is a pair of numbers where the smaller appears after the larger in the permutation.)

**Answer:** 22505

*Proposed by Aaryan Vaishya*

Note that if a permutation has $i$ inversions, $i^2 = i + 2\binom{i}{2}$. Thus, $\mathbb{E}[i^2] = \mathbb{E}[i] + 2\mathbb{E}[\binom{i}{2}]$. Now, we first find the expected number of inversions. Each pair of values $(a, b)$ are inverted in half of the permutations, so $\mathbb{E}[i] = \frac{1}{2}\binom{2023}{2}$.

Now, we count pairs of inversions. First, the expected number of inverted pairs $(a, b)$ and $(c, d)$ with all distinct values is $\frac{3}{4}\binom{2023}{4}$. Then, if we have $a < b < c$, the 6 possible orderings and the number of pairs of inversions between them are: *abc* with 0, *acb* with 0, *bac* with 0, *bca* with 1, *cab* with 1, and *cba* with 3. So, the expected number of inverted pairs with shared values is $\frac{5}{6}\binom{2023}{3}$.

Summing now gives $\mathbb{E}[i^2] = \frac{1}{2}\binom{2023}{2} + \frac{3}{2}\binom{2023}{4} + \frac{5}{3}\binom{2023}{3} = \frac{4602832}{9}\binom{2023}{2}$. Now, we can find the prime divisors of $m$ are 2, 7, 13, 17, 337, 22109, giving an answer of $\boxed{22505}$.

**27.** Let $S$ be the sum of

$$\binom{a_1}{1000} + \binom{a_2}{1000} + \cdots + \binom{a_{10}}{1000}$$

over all tuples $(a_1, a_2, \ldots, a_{10})$ of nonnegative integers such that $a_1 + a_2 + \cdots + a_{10} = 2023$. Find the largest nonnegative integer $k$ for which $2^k$ divides $S$.

**Answer:** 11

*Proposed by Darsh Patel*

It follows from symmetry that this is 10 times the sum of $\binom{a_1}{1000}$ over all such $a_1, a_2, \ldots, a_{10}$.

For a particular value of $k$ for $a_1$, there are $\binom{2031-k}{8}$ solutions to $a_2 + a_3 + \cdots + a_{10} = 2023 - k$. Then this value of $k$ contributes $\binom{k}{1000}\binom{2031-k}{8}$ to the sum. So, we wish to compute

$$\sum_{k=0}^{2023} \binom{k}{1000}\binom{2031-k}{8}$$

However, this is just the coefficient of $x^{2031}$ in

$$\begin{aligned}
&\left(x^0\binom{0}{1000} + x^1\binom{1}{1000} + \cdots\right)\left(x^0\binom{0}{8} + x^1\binom{1}{8} + \cdots\right) \\
&= \frac{x^{1000}}{(1-x)^{1001}} \cdot \frac{x^8}{(1-x)^9} \\
&= \frac{x^{1008}}{(1-x)^{1010}} \\
&= x^{1008}\left(\binom{1009}{1009}x^0 + \binom{1010}{1009}x^1 + \ldots\right)
\end{aligned}$$

which is $\binom{2032}{1009}$.

We now compute

$$\nu_2\left(10\binom{2032}{1009}\right) = \nu_2(2032!) - \nu_2(1009!) - \nu_2(1023!) + 1 = 2025 - 1002 - 1013 + 1 = 11$$

So, the desired value is $\boxed{11}$.

**28.** Consider reals $a_1, a_2, \ldots, a_5 \geq 0$ with a sum of 1. Define $P$ and $Q$ as follows:

$$\begin{aligned}
P &= \min\{a_1, a_2\} + 2 \cdot \min\{a_2, a_3\} + \cdots + 5 \cdot \min\{a_5, a_1\} \\
Q &= \max\{a_1, a_2\} + 2 \cdot \max\{a_2, a_3\} + \cdots + 5 \cdot \max\{a_5, a_1\}
\end{aligned}$$

When $P$ reaches its maximum, the range of all possible values of $Q$ is $[a, b]$. Find $\lfloor 100(a+b) \rfloor$.

**Answer:** 733

*Proposed by Andrew Xing*

First we prove that $a_2 \leq a_3 \leq a_4 \leq a_5$. If this is false for some $a_i$, if $a_i > a_{i+1}$, we can subtract a small amount, $d$, from $a_i$ and add it to $a_{i+1}$. $\min\{a_i, a_{i+1}\}$ will increase by $d$ and $\min\{a_{i-1}, a_i\}$ will decrease by at most $d$, so $P$ increases.

Now, replacing $(a_1, a_5)$ by $(\frac{a_1+a_5}{2}, \frac{a_1+a_5}{2})$ will increase $P$ if $a_1 \neq a_5$, so they must be equal. Further, considering $(a_4 + 2d, a_5 - d, a_1 - d)$, we see that $a_4 = a_5$ as well.

Thus, we are left with $a_2 = x$, $a_3 = y$, and $a_1 = a_4 = a_5 = z$, where $x \leq y \leq z$. We have $P = x + 2x + 3y + 4z + 5z = 3x + 3y + 9z$ and $x + y + 3z = 1$, so $\max P = 3$ when these conditions are satisfied. Additionally,

$$Q = z + 2y + 3z + 4z + 5z = 2y + 13z = 2(y + 3z) + 7z = 2(1 - x) + 7z.$$

This is maximized when $x = 0, z = \frac{1}{3}$, and minimized when $x = \frac{1}{5}, y = \frac{1}{5}$, yielding a range of $[3, \frac{13}{3}]$, and our final answer is $\boxed{733}$.

**29.** Find the number of distinct values that

$$\frac{i^4 - 1}{i^2}$$

takes modulo 2023 over all $i$ such that $\gcd(i, 2023) = 1$.

**Answer:** 159

*Proposed by Jessica Wan*

Suppose for two distinct values $a, b$, $\frac{a^4-1}{a^2} \equiv \frac{b^4-1}{b^2}$. Then,

$$b^2(a^4 - 1) \equiv a^2(b^4 - 1) \implies 2023 \mid (a^2 - b^2)(a^2b^2 + 1)$$

Note that we can consider the value modulo 7 and $17^2$ separately. As $-1$ is not a quadratic residue of any prime $\equiv 3 \pmod 4$, if $7 \mid (a^2 - b^2)(a^2b^2 + 1)$, $a \equiv b, -b \pmod 7$. Thus, the expression takes $6/2 = 3$ distinct values modulo 7.

Considering modulo 289, we have the following cases:

- $17^2 \mid a - b \implies a \equiv b \pmod{17^2}$
- $17^2 \mid a + b \implies a \equiv -b \pmod{17^2}$
- $17^2 \mid 1 + a^2b^2 \implies ab \equiv 38, 251 \pmod{17^2}$
- $17 \mid\mid a - b$ and $17 \mid 1 + a^2b^2$. In this case, $a^4 \equiv -1 \pmod{17}$, so $a \equiv 2, 8, 9, 15 \pmod{17}$.
- $17 \mid\mid a + b$ and $17 \mid 1 + a^2b^2$. Again, we find $a \equiv 2, 8, 9, 15 \pmod{17}$.

Now, the above tells us that all $i = 17k \pm 2$ give one value and all $i = 17k \pm 8$ give another. For other moduli, note that groups of 4 are formed by $a$, $-a$, $\frac{38}{a}$, and $\frac{251}{a}$. The values in each group provide one additional, distinct modulus.

Thus, we have $2 + \frac{17(16-4)}{4} = 53$ total possible moduli for 289. Finally, there are $3 \cdot 53 = \boxed{159}$ distinct values of $\frac{i^4-1}{i^2}$.

**30.** Consider cyclic quadrilateral $ABCD$ with integer side lengths. Let $F = AC \cap BD$, and denote the circumcenters of $CBF, ABCD, DCF$ as $H, I, J$, respectively. Given that $AB = AD = 6$, $AF = 3$, and $CF = 9$, the area of $\triangle HIJ$ can be expressed in simplest form as $\frac{p\sqrt{q}}{r}$. Compute $100p + 10q + r$.

**Answer:** 2270

*Proposed by Tom Zhang*

Note that $AF \cdot AC = AB^2, \triangle ABF \sim \triangle ACB, CB = 2BF$. By fact 5, $AC$ bisects $\angle BCD, CD = 2DF$. Thus, we can let $BF = x, BC = 2x; FD = y, CD = 2y$. Stewart has $(2x)^2y + (2y)^2x = 4xy(x+y) = (x+y)(xy+81), xy = 27, (2x)(2y) = 108 = BC \cdot DC$. As $BC, DC$ have integer sidelengths which also have to satisfy the triangle inequality, we can attain $BC = 9, DC = 12$(WLOG, we let $BC < DC$)

Then, I claim lines $BH, DJ$ meet the circumcircle of $ABCD$ at $P$ and $C, H, I, J, P$ are concyclic. As $AB^2 = AF \cdot AC$, we can conclude that $AB$ is tangent to the circumcircle of $\triangle CBF$. Let line $BH$ meet the circumcircle of $ABCD$ at $P', \angle ABP' = 90^\circ$, which means $AP'$ is the diameter of $(ABCD)$. Since $ABCD$ is a kite, which means line $AI$ is the diameter of its diameter, so $P'$ lies on $AI$. A similar reason tells $P'$ lies on $DJ$ which means $P' = P$.

As $\triangle BCF \sim \triangle ACD, \angle HCF = \frac{\alpha}{2} = \angle ACI, \angle HCI = \alpha = \angle HPI$. We can conclude $H, I, P, C$ lie on the same circle. As $I$ lies on the perpendicular bisector $CD$ so as $J$, $IJ \perp CD$. By the same reason, we have $IH \perp BC$. In $\triangle ABC, \angle BAC = 180 - (90 + \frac{\alpha}{2} + \alpha) = 90 - \frac{3\alpha}{2}, \angle CAP = \frac{\alpha}{2}$. Since $IA = IC, \angle CIP = \alpha$ As $CADP$ is a cyclic quadrilateral, $\angle CDP = \angle CAI = \frac{\alpha}{2}$, Since $JD = JC, \angle CJP = \alpha$ yielding $ICPJ$ is a cyclic quadrilateral. Thus, the pentagon $HIJPC$ is cyclic. Since $\angle HPI = \angle JPI, HI = IJ$ as desired.

Now we come back for computation, the $C-$ altitude of $\triangle BCD$ is $\frac{9\sqrt{15}}{4}$. By $R = \frac{abc}{4A}$, we have $CH = \frac{6\sqrt{15}}{5}, CI = \frac{8\sqrt{15}}{5} = CJ$. $\cos\alpha = \frac{12^2+9^2-6^2}{2\cdot 12\cdot 9} = \frac{7}{8}, \cos 2\alpha = \cos\angle HCJ = \frac{17}{32}, \sin\angle HCJ = \sin\angle HIJ = \frac{7\sqrt{15}}{32}, HJ = \frac{7\sqrt{15}}{5}$. By Ptolemy in $HIJC, HI = IJ = \frac{4\sqrt{15}}{5}$. The desired area is then $(\frac{4\sqrt{15}}{5})^2 \cdot \frac{7\sqrt{15}}{32} \cdot \frac{1}{2} = \frac{21\sqrt{15}}{20} \implies \boxed{2270}$.

Alternately, after finding $BC = 9, DC = 12$, one can notice that $\angle BFC = \angle CDA = 180^\circ - \angle CFD$, so $\triangle CHB \sim \triangle CIA \sim \triangle CJD$, and there is a spiral similarity with center $C$ and ratio $\frac{CI}{CA}$ that takes $\triangle BAD$ to $\triangle JIH$.

Thus, to find the area of $\triangle JIH$, note $BA = AD = 6$ and $BD = \frac{BC+CD}{2} = \frac{21}{2}$. $[BAD] = \frac{63\sqrt{15}}{16}$. Now, $\angle ICA = 90 - \angle CDA$. As $\triangle CDA$ is isosceles with $CD = CA = 12$, we find $\sin CDA = \frac{\sqrt{15}}{4}$ and $\frac{CI}{CA} = \frac{2}{\sqrt{15}}$.

Thus, $[HIJ] = [BAD] \cdot \frac{4}{15} = \frac{21\sqrt{15}}{20}$, giving an answer of $\boxed{2270}$.