Answer Key

Algebra and Number Theory

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2. 110
3. 36
4. 9
5. 1426
6. 36

Geometry

1. 10
2. 15
3. 54
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Algebra and Number Theory

1. If the value of x satisfying

$$2(5(2(5(2(5x-4)+3)-4)+3)-4)+3=x$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m+n$.

Answer: 14

Proposed by Aaron Hu

This is a linear equation, so there is only one solution that must satisfy

$$x=2(5x-4)+3=10x-5 \implies x=\frac{5}{9},$$

and the answer is $5+9=\boxed{14}$.

2. Compute the sum of all reals a such that there exists a quadratic P with leading coefficient 1 satisfying $P(a)=10$, $P(10)=100$, $P(100)=a$.

Answer: 110

Proposed by Aaron Hu

Since P is a monic quadratic, $Q(x)=P(x)-x^2$ is linear. Then $Q(a)=10-a^2$, $Q(10)=0$, $Q(100)=a-100^2$, so

$$\frac{10-a^2}{a-10}=\frac{Q(a)-Q(10)}{a-10}=\frac{Q(100)-Q(10)}{100-10}=\frac{a-100^2}{90}.$$

Cross multiplying and simplifying gives $91a^2-(100^2+10)a+c=0$ for some constant c, so from Vieta, the answer is $\frac{100^2+10}{91}=\boxed{110}$. Alternatively, directly assigning variables to the coefficients of P and solving the system of equations works.

3. Find the smallest positive integer x such that $\binom{5043}{7}+x$ is a perfect square.

Answer: 36

Proposed by Jessica Wan and Derek Liu

We substitute $5040=7!=n$ to rewrite $\binom{5043}{7}=(n+3)(n+2)(n+1)(n-1)(n-2)(n-3)=(n^2-1)(n^2-4)(n^2-9)=n^6-14n^4+49n^2-36=(n(n^2-7))^2-36$. Thus, $x=\boxed{36}$ is a valid solution, and it is the minimum as $(n(n^2-7))^2-(n(n^2-7)-1)^2=2n(n^2-7)-1>36$.

4. On the graph of $y=x^4-10x^3+15x^2+10x-6$, two distinct points $P=(1,10)$ and Q, both with integer coordinates, are marked. If the distance PQ is also an integer, find the sum of all possible x-coordinates of Q.

Answer: 9

Proposed by Jessica Wan

Claim: $\overline{PQ}$ must be parallel to the x-axis. Let $P=(a,f(a))$, $Q=(b,f(b))$. Then,

$$PQ=\sqrt{(a-b)^2+(f(a)-f(b))^2}=(a-b)\cdot\sqrt{1+\left(\frac{f(a)-f(b)}{a-b}\right)^2}$$

Thus, as $a-b \mid f(a)-f(b)$, if $f(x)$ has integer coefficients, the only way for PQ to be an integer is if the y-coordinates of P, Q are equal.

Finding roots of $x^4 - 10x^3 + 15x^2 + 10x - 6 = 10$, $-1, 2, 8$ are the possible x-coordinates of Q for a desired answer of $\boxed{9}$.

5. If $425^2 + 312^2 = 520^2 + 87^2$ can be written as the product of two distinct primes, what is their sum?

Answer: 1426

Proposed by Jessica Wan

Note that both primes must be $1 \pmod 4$, and by Fermat's Christmas Theorem they can be written as $p = a^2 + b^2$, $q = c^2 + d^2$ uniquely up to ordering. Now, it is known that the product of these two primes pq can be expressed as the sum of two squares in exactly two distinct ways:

$$(ac - bd)^2 + (ad + bc)^2 = (ad - bc)^2 + (ac + bd)^2$$

Note that $ac - bd \equiv ac + bd \pmod 2$ and $ad + bc \equiv ad - bc \pmod 2$. Thus, wlog, $ac - bd = 312$, $ac + bd = 520 \implies ac = 416$, $bd = 104$ and $ad + bc = 425$, $ad - bc = 87 \implies ad = 256$, $bc = 169$.

So, we find $a = 32$, $b = 13$, $c = 13$, $d = 8$, and $p = 32^2 + 13^2 = 1193$, $q = 13^2 + 8^2 = 233$, giving the desired answer of $\boxed{1426}$.

(Note: original form of this problem used the primes $p = 1201 = 24^2 + 25^2$ and $q = 233$.)

6. The smallest positive solution of the equation below can be written as $a\pi/b$, with a, b relatively prime positive integers. Find $10a + b$.

$$\tan\left(\frac{\pi}{6} + x\right)\tan\left(\frac{\pi}{6} + 3x\right)\tan\left(\frac{\pi}{6} + 9x\right) = \tan\left(\frac{\pi}{3} + x\right)\tan\left(\frac{\pi}{3} + 3x\right)\tan\left(\frac{\pi}{3} + 9x\right).$$

Answer: 36

Proposed by Jessica Wan

First, observe that $\tan\left(\frac{\pi}{3} + y\right) = 1/\tan\left(\frac{\pi}{6} - y\right)$ for each $y \in \mathbb{R}$. Next, note that (by the Tangent Addition and Subtraction Formulas):

$$\tan\left(\frac{\pi}{6} - y\right)\tan\left(\frac{\pi}{6} + y\right) = \frac{\frac{1}{\sqrt{3}} + \tan y}{1 - \frac{\tan y}{\sqrt{3}}} \cdot \frac{\frac{1}{\sqrt{3}} - \tan y}{1 + \frac{\tan y}{\sqrt{3}}} = \frac{\frac{1}{3} - \tan^2 y}{1 - \frac{\tan^2 y}{3}}$$

$$= \frac{1}{\tan y} \cdot \frac{\tan y - 3\tan^3 y}{3 - \tan^2 y} = \frac{\tan 3y}{\tan y},\ \forall y \in \mathbb{R},$$

where we used the Tangent Triple Angle Formula in the last step. Using the above two observations, we can rewrite the given equation as:

$$\frac{\tan 3x}{\tan x} \cdot \frac{\tan 9x}{\tan 3x} \cdot \frac{\tan 27x}{\tan 9x} = 1 \iff \tan 27x = \tan x,$$

which has minimal solution $x = \frac{\pi}{26}$, so our answer is $\boxed{36}$.

Alternate solution from Elliott Liu: Let $\omega = e^{\frac{2\pi i}{6}}$ and $\alpha = e^{2ix}$. The given condition is equivalent to

$$\frac{\omega\alpha - 1}{\omega\alpha + 1} \cdot \frac{\omega\alpha^3 - 1}{\omega\alpha^3 + 1} \cdot \frac{\omega\alpha^9 - 1}{\omega\alpha^9 + 1} = \frac{\omega^2\alpha - 1}{\omega^2\alpha + 1} \cdot \frac{\omega^2\alpha^3 - 1}{\omega^2\alpha^3 + 1} \cdot \frac{\omega^2\alpha^9 - 1}{\omega^2\alpha^9 + 1}.$$

The key observation is that $(\omega t-1)(\omega^2 t+1) = -t^2+t-1$ and $(\omega t+1)(\omega^2 t-1) = -t^2-t-1$, so the condition above reduces to

$$(-\alpha^2+\alpha-1)(-\alpha^6+\alpha^3-1)(-\alpha^{18}+\alpha^9-1) = (-\alpha^2-\alpha-1)(-\alpha^6-\alpha^3-1)(-\alpha^{18}-\alpha^9-1).$$

Now, telescoping yields

$$\frac{\alpha^{27}+1}{\alpha+1} = \frac{\alpha^{27}-1}{\alpha-1},$$

whence $\alpha^{27} = \alpha$, so the smallest x is attained when $\alpha = e^{\frac{2\pi i}{26}}$ and our answer is $10+26 = 36$.

Geometry

1. What is the sum of all possible values of the number of intersections between a circle of radius 3 and an $8-15-17$ right triangle?

Answer: 10

Proposed by Alex Li

Note that the inradius of the triangle is 3, so the circle only intersects all three sides when it is the incircle. In all other cases, the circle intersects at most two sides. We can construct the cases for $1, 2, 3$, and 4 intersections pretty easily, and the answer is $\boxed{10}$.

2. Let $ABCD$ be a parallelogram with $\angle A > 90^\circ$. If the angle bisector of $\angle ABC$ bisects $\overline{AD}$ and $AB = 20$, find the number of possible integer values of length BD.

Answer: 15

Proposed by Aaron Hu

Let M denote the midpoint of $\overline{AD}$ and $X = \overline{BM} \cap \overline{AC}$. From Angle Bisector Theorem and similar triangles,

$$\frac{AB}{BC} = \frac{AX}{XC} = \frac{AM}{BC} = \frac{AM}{AD} = \frac{1}{2},$$

so $BC = AD = 40$. Clearly, $BD < AB + AD = 60$, and since $\angle A > 90^\circ$, we have

$$BD > \sqrt{AB^2+AD^2} = \sqrt{20^2+40^2} = \sqrt{2000} \implies BD \geq 45.$$

Then the answer is $60-45 = \boxed{15}$.

3. Externally tangent circles ω_1, ω_2, each with radius 1, are inscribed in equiangular hexagon $ABCDEF$ such that ω_1 is tangent to EF, FA, AB and ω_2 is tangent to AB, BC, CD. Suppose that $AB = CD = EF$ and $BC = DE = AF$. Given that the area of $ABCDEF = a + b\sqrt{c}$ for positive integers a, b, c, where c is not divisible by the square of any prime, find $a^2+b^2+c^2$.

Answer: 54

Proposed by Alex Li

Drop perpendiculars from the centers of ω_1 and ω_2 to side AB. Note that angle FAB is bisected by the line connecting A and the center of ω_1, forming a $30-60-90$ triangle. Therefore, we can find that $AB = 2+\frac{2}{\sqrt{3}}$. Similarly, we can find $BC = \frac{2}{\sqrt{3}}$.

Filling out the hexagon into a $\left(2+\frac{4}{\sqrt{3}}\right) \times (2+\sqrt{3})$ rectangle and subtracting off the residual triangles gives area

$$\frac{2}{\sqrt{3}}(2+\sqrt{3})^2 - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}(\sqrt{3}+1)^2 = 6+3\sqrt{3},$$

so the answer is $36 + 9 + 9 = \boxed{54}$.

4. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$, and let $E = \overline{AC} \cap \overline{BD}$. If $AB = 7$, $CD = 8$, and $\odot(CDE)$ bisects $\overline{AD}$, what is the perimeter of $ABCD$?

Answer: 71

Proposed by Aaron Hu

Let $x = AC = BD, y = BC = AD$. If we let M denote the midpoint of $\overline{AD}$, then

$$\frac{1}{2}y^2 = AM \cdot AD = AE \cdot AC = \frac{AB}{AB + CD} \cdot AC^2 = \frac{7}{15}x^2,$$

where we used Power of a Point and similar triangles. Since isosceles trapezoids are cyclic, we have from Ptolemy's theorem that

$$y^2 + 56 = AB \cdot CD + BC \cdot AD = AC \cdot BD = x^2 = \frac{15}{14}y^2,$$

so $y = \sqrt{14 \cdot 56} = 28$. Then the answer is $7 + 8 + 2 \cdot 28 = \boxed{71}$.

5. Consider rhombi $PQAB$ and $PQCD$ where $\overline{AB}, \overline{CD}$ are on opposite sides of PQ. If $PQ = 40$, $AD = 39$, and the total area of the two rhombi is maximized, find BC.

Answer: 89

Proposed by Aaron Hu

Note that $AB = CD$ and $AB \parallel PQ \parallel CD$, so $ABDC$ is a parallelogram. Additionally, $PB = PD = QA = QC$ and $\angle BPD = \angle AQC$ from the parallel sides, so $\triangle BPD \cong \triangle AQC$. Then the total area of the two rhombi is equal to the area of the parallelogram. We have

$$[ABCD] = AB \cdot AD \cdot \sin\angle BAD = PQ \cdot AD \cdot \sin\angle BAD,$$

which is maximized when $\angle BAD = 90^\circ$. Thus, $BC = \sqrt{(AB + CD)^2 + AD^2} = \sqrt{80^2 + 39^2} = \boxed{89}$.

6. Cyclic pentagon $ABCDE$ has $AB \parallel DE$. The line through C parallel to AB intersects BD at F. Given $AB = 11$, $BD = 20$, $DE = 13$, and $\frac{BC}{CE} = \frac{1}{2}$, $\frac{BF}{FD} = \frac{m}{n}$ in simplest terms. Find $m + n$.

Answer: 16

Proposed by Jessica Wan and Elliott Liu

Let $BC \cap DE$ be X. The desired ratio equals $\frac{BC}{CX}$.

First, as $ABDE$ is an isosceles trapezoid, we can find $\cos BDE = \frac{1}{20}$ and $BE = \sqrt{543}$. As $\angle BCE = \angle BDE$, LOC on $\triangle BCE$ gives: $543 = BC^2 + CE^2 - 2 \cdot BC \cdot CE \cdot \frac{1}{20}$. Thus, $BC = \sqrt{\frac{905}{8}}$.

Denote the above value as $BC = a$. Let $XC = x$, $XD = y$. The similarity between $\triangle XCE$ and $\triangle XDB$ gives

$$\frac{x}{y} = \frac{y + 13}{x + a} = \frac{a}{10}$$

Substitute $y = \frac{10x}{a}$ to find $\frac{10x + 13a}{a(x+a)} = \frac{a}{10} \implies x = \frac{a(130 - a^2)}{a^2 - 100}$. The desired ratio is $\frac{a}{x} = \frac{a^2 - 100}{130 - a^2} = \frac{7}{9}$, for an answer of $\boxed{16}$.

Combinatorics

1. Tiger has a coin that has an unknown probability of flipping heads. Andrew then tells Tiger that the probability of flipping heads is a random fraction in the infinite set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$. Given that Tiger gets three heads after flipping the coin three times and the probability of the coin being fair is $p = \frac{m}{n}$ for relatively prime positive integers m and n, find $10m + n$.

Answer: 134

Proposed by Alex Li

The desired probability is

$$\frac{\frac{1}{2^3}}{1^3 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots} = \frac{7}{64}.$$

The answer is $\boxed{134}$.

2. In $\triangle ABC$, we have $AB = BC = 1$ and $AC = \sqrt{2} \approx 1.4142$. Points P and Q are chosen uniformly on $\overline{AB}$ and $\overline{BC}$, respectively. Let p be the probability that the perimeter of $ACQP$ is less than BPQ. Compute $\lfloor 1000p \rfloor$.

Answer: 42

Proposed by Karthik Vedula

Let $BP = a$ and $BQ = b$. The condition is satisfied if and only if

$$\sqrt{a^2 + b^2} + (1 - a) + (1 - b) + \sqrt{2} < \sqrt{a^2 + b^2} + a + b \implies a + b > 1 + \frac{\sqrt{2}}{2}.$$

Using geometric probability, we have $p = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)^2 = \frac{3}{4} - \frac{\sqrt{2}}{2}$, and using the approximation, we get $\lfloor 1000p \rfloor = \lfloor 750 - 1414.2/2 \rfloor = \boxed{42}$.

3. Alex is practicing basketball for 100 days. He makes a layup every x days, starting on Day 1, and makes a 3-pointer every y days, starting on Day 2. How many ordered pairs of integers (x, y) with $1 \leq x, y \leq 10$ are there such that there will be a day where he makes a layup and a 3-pointer?

Answer: 63

Proposed by Aaron Hu

There must exist nonnegative integers a, b such that $1 + ax = 2 + by$. This rearranges to $ax - by = 1$, so $\gcd(x, y) = 1$. Whenever x, y are relatively prime, there exists a positive integer $n \leq xy \leq 100$ such that $n \equiv 1 \pmod{x}$ and $n \equiv 2 \pmod{y}$, implying that all such (x, y) work.

We will use complimentary counting to find all relatively prime pairs (x, y). There are $5^2 + 3^2 + 2^2 + 1^2 = 39$ pairs where both x, y are divisible by 2, 3, 5, and 7, respectively. However, this double counts when $(x, y) = (6, 6), (10, 10)$, so the final answer is $100 - (39 - 2) = \boxed{63}$.

4. Compute the last three digits of the number of tuples of integers $(a_1, a_2, \ldots, a_6)$ such that $0 \leq a_i \leq 11$ and $a_1 + 2a_2 + \cdots + 6a_6$ is divisible by 7.

Answer: 570

Proposed by Aaryan Vaishya

Note that the number formed by $\overline{a_6 a_5 \cdots a_1}$ in base 12 is congruent to $a_1 + 5a_2 + 4a_3 + 6a_4 + 2a_5 + 3a_6 \pmod{7}$. Thus, rearranging the digits a_i in the proper order, we find a bijection between the

desired tuples and multiples of seven under 12^6. Our answer is then the last three digits of $\lceil \frac{12^6}{7} \rceil = \lceil \frac{(20\cdot 7+4)^3}{7} \rceil = 20^3 \cdot 7^2 + 3 \cdot 20^2 \cdot 7 \cdot 4 + 3 \cdot 20 \cdot 4^2 + 9 \equiv \boxed{569} \pmod{1000}$.

5. A bird wants to walk from $(0,0)$ to $(12,12)$ using only unit steps going right and up. Suitable paths do not pass through any point with odd x and y coordinates that sum to 12. How many more suitable paths are there than unsuitable paths?

Answer: 924

Proposed by Jessica Wan

Substitute $2n = 12$. Let A be the number of suitable paths, and B be the complement. Any path passes through exactly one point on $x + y = 12$. Note $A + B = \binom{4n}{2n}$, the coefficient of x^{2n} in $(1+x)^{4n}$.

$A = \binom{2n}{0}\binom{2n}{2n} + \binom{2n}{2}\binom{2n}{2n-2} + \cdots + \binom{2n}{2n}\binom{2n}{0}$

$B = \binom{2n}{1}\binom{2n}{2n-1} + \binom{2n}{3}\binom{2n}{2n-3} + \cdots + \binom{2n}{2n-1}\binom{2n}{1}$

Inspired by the expression for $A + B$, we can see that $A - B$ is the coefficient of x^{2n} in $(x-i)^{2n}(x+i)^{2n} = (x^2+1)^{2n}$ if n is even. Then, $A - B = \binom{2n}{n}$, for an answer of $\binom{12}{6} = \boxed{924}$.

6. Determine the number of ways to permute AAaaBBbb so that:

- the number of A/a's between the two B's is even,
- the number of A/a's between the two b's is odd,
- the number of B/b's between the two A's is even,
- the number of B/b's between the two a's is odd.

For example, abBBabAA is a valid permutation.

Answer: 288

Proposed by Jessica Wan and Elliott Liu

Consider an arrangement of unspecified-case a's and b's. In particular, consider the parity of the number of b's between consecutive a's. We find the only valid arrangements are:

$$A\boxed{E}A\boxed{O}A\boxed{O}A\ ,\ A\boxed{E}A\boxed{E}A\boxed{O}A\ ,\ A\boxed{O}A\boxed{E}A\boxed{E}A\ ,\ A\boxed{O}A\boxed{O}A\boxed{E}A$$

We ignore the latter two for now, as they are symmetric to the first two. For the first form, the valid arrangements of a/A's are: $aAaA$, $AAaa$, $AaaA$. For the second, they are: $AAaa$, $AaAa$, $aAAa$.

Thus, the number of solutions where the a/A's are arranged in orders ($aaAA$ or $AAaa$), ($aAAa$ or $AaaA$), and ($aAaA$ or $AaAa$) are equal for each distribution of a/A's and b/B's. We find a similar fact for the arrangement of b/B's.

So, we merely need to find the number of solutions with a/A's ordered as ($aaAA$ or $AAaa$) and b/B's ordered as ($bbBB$ or $BBbb$), and multiply this by 9.

With orders $aaAA$ and $bbBB$, we find the valid orderings are: $babBBaAA$, $babaBBAA$, $babaBAAB$, $babaABBA$, $babaAABB$, $ababBBAA$, $ababBAAB$, $ababABBA$, $ababAABB$, $abaAAbBB$.

With orders $aaAA$ and $BBbb$, we find the valid orderings are: $BBbabaAA$, $BBababAA$, $BBabaAAb$, $aBBbabAA$, $aBBbaAAb$, $aBaABbAb$.

The cases with $AAaa$ are symmetric. Thus, we have $9 \cdot 2 \cdot (10+6) = \boxed{288}$ possible permutations.