Answer Key

1. 14
2. 4
3. 169
4. 6079
5. 2520
6. 132
7. 370
8. 106050
9. 24
10. 12

1. There is a world where people are either truth tellers who always tell the truth, liars who always lie, or alternators who change the truth value of their responses every time they respond. A group of people were each asked if they were a truth teller, and then asked if they were an alternator. If 75 people said yes to being a truth teller and 61 people said yes to being an alternator, how many truth tellers were in the group?

Answer: 14

Proposed by Kevin Liu

Note that the truth tellers always reply yes-no-no, the liars always reply yes-yes-no, and the alternators always reply yes-yes-yes or no-no-no. Then everyone said yes to the first question, and only the truth tellers said no to the second question, so the answer is $75 - 61 = \boxed{14}$.

2. Aaron stands blindfolded near the edge of a cliff so that if he takes one step right he will fall off. Because he likes living on the edge, every minute he moves to the right one step with probability $\frac{3}{4}$ and moves to the left two steps with probability $\frac{1}{4}$. What is the expected number of minutes Aaron needs to fall off the cliff?

Answer: 4

Proposed by Andrew Xing

Call E the expected number of minutes for Aaron to move one step right from where he starts. We are looking for E, as this equals the expected number of minutes to fall off the cliff. We have the equation $E = 3E \cdot \frac{1}{4} + 1$, as he will fall after the first move with $\frac{3}{4}$ chance or move two steps to the left with $\frac{1}{4}$ chance, in which case he has an expected $3E$ minutes to fall. Thus, $E = \boxed{4}$.

Alternatively, Aaron moves to the right an average of $\frac{3}{4} - 2 \cdot \frac{1}{4} = \frac{1}{4}$ of a step every minute, so he needs an expected $\boxed{4}$ minutes to fall.

3. Consider $\triangle ABC$ with $AB = 13$, $BC = 14$, $CA = 15$. Let M be the midpoint of the arc BC containing A in the circumcircle of $\triangle ABC$. If X is the unique point on segment AC with $MA = MX$, find CX^2.

Answer: 169

Proposed by Aaryan Vaishya

Construct the point Y on segment CA such that $CY = AB = 13$. Then, as $MC = MB$, $CY = BA$, and $\angle MCY = \angle MCA = \angle MBA$, triangles MCY and MBA are congruent. Thus, $MY = MA$ and Y is the desired point X, so our answer is $CX^2 = \boxed{169}$.

4. Find the sum of all reals x such that $\sqrt[3]{2024 - x} + \sqrt{x - 2023} = 1$.

Answer: 6079

Proposed by Sailalitha Kodukula

Substitute $\sqrt{x - 2023} = y$. Then, $\sqrt[3]{1 - y^2} = 1 - y$, $1 - y^2 = 1 - 3y + 3y^2 - y^3$, $y^3 - 4y^2 + 3y = 0$. This gives $y = 0, 1, 3$ or $x - 2023 = 0, 1, 9$, all of which give valid solutions. Thus, the sum of all real solutions x is $3 \cdot 2023 + 0 + 1 + 9 = \boxed{6079}$.

5. A magical ant is located at the origin. Every second, it can teleport a positive integer distance in either the positive x, y, or z direction, but cannot teleport in the same direction twice in a row. Find the number of ways the ant can get to $(2, 3, 5)$.

Answer: 2520

Proposed by Darsh Patel

Call such a walk from $(0,0,0)$ to $(2,3,5)$ an *ant-walk*. Now, consider walks from $(0,0,0)$ to $(2,3,5)$ by taking steps of one unit in the positive x, y, or z direction, and call such a walk a *unit-walk*.

We consider a bijection from unit-walks to ant-walks: for every run of consecutive unit steps in the same direction, turn them into one single ant-step. So, for every unit-walk there is a unique corresponding ant-walk. It follows that there are the same number of each. Our answer is simply the number of unit walks, $\binom{10}{2,3,5} = \boxed{2520}$.

6. Let $P(x)$ be a monic quadratic polynomial with positive integer coefficients satisfying

$$n+2 \mid P(n) + P(n+1)$$

for all positive integers n. Compute the minimum possible value of $P(10)$.

Answer: 132

Proposed by Tom Zhang

Since this holds for infinitely many positive integers, $x+2$ must be a factor of $P(x)+P(x+1)$. If we let $P(x) = x^2 + ax + b$, we require

$$0 = P(-2) + P(-1) = (4 - 2a + b) + (1 - a + b) = 5 - 3a + 2b,$$

so $(a,b) = (2k+3, 3k+2)$ for nonnegative integer k. Clearly, $P(10)$ is minimized when $k = 0$, so the answer is $10^2 + 3(10) + 2 = \boxed{132}$.

7. Suppose Maximillian the monkey has an infinite supply of bananas. Every 25 seconds, Maximillian eats a banana either in one or two bites. However, if he ever eats three bananas in a row in three bites, Maximillian will throw up and falls asleep. Let $\mathbb{E}$ be the expected number of total bites Maximillian takes before he throws up. If the probability Maximillian eats a banana in one bite is $\frac{3}{5}$, $\mathbb{E}$ can be written in simplest form as $\frac{m}{n}$. Find $m+n$.

Answer: 370

Proposed by Jayden Lang

Denote the expected value of total bites Maximillian takes before throwing up after eating n bananas in a row in one bite by E_n. If we visualize this with a states diagram, we can quickly formulate a system of equations in terms of E_0, E_1, E_2 and E_3:

$$E_0 = \frac{2}{5}(E_0 + 2) + \frac{3}{5}(E_1 + 1)$$

$$E_1 = \frac{2}{5}(E_0 + 2) + \frac{3}{5}(E_2 + 1)$$

$$E_2 = \frac{2}{5}(E_0 + 2) + \frac{3}{5}(E_3 + 1)$$

$$E_3 = 0$$

Solving this, we find $\mathbb{E} = E_0 = \frac{343}{27}$, $m+n = \boxed{370}$.

8. A country has 101 cities, each pair connected by a two-way flight by either Company A or B. If 30 of the flights from each city are supplied by Company A, how many triples of cities are there such that among the 3 direct flights between them, there are flights by both companies?

Answer: 106050

Proposed by Jessica Wan

Let n be the number of triples that satisfy the conditions. Let m be the complement. $m+n=\binom{101}{3}$.

Now, for each city C, count the pairs of cities whose flights to C are of the same company (ie. ordered triples $(C; D, E)$ such that $C-D$ and $C-E$ are by the same company). There are $\binom{30}{2}+\binom{70}{2}$ such triples for each C. Note each triple in m is counted 3 times and each in n once.

So, $3m+n=101\cdot(\binom{30}{2}+\binom{70}{2})$. Thus, $n=(3(m+n)-(3m+n))/2=\boxed{106050}$.

Alternatively, consider any city C. Then there are $30\cdot 70=2100$ pairs of cities D, E such that $C-D$ is Company A and $C-E$ is company B. Note that exactly one of D, E also has this property with the other two vertices, so summing over all cities and dividing by 2 for double counting gives $101\cdot 2100/2=\boxed{106050}$.

9. Isosceles trapezoid $ABCD$ with $\overline{AB}\parallel\overline{CD}$ has an inscribed circle ω with center I. Given $BC=50$ and $IB=30$, find the radius of ω.

Answer: 24

Proposed by Yuhan Niu

Since $\overline{AB}\parallel\overline{CD}$, we have

$$\angle BIC=\angle 180^\circ-(\angle ICB+\angle IBC)=180^\circ-\frac{1}{2}(\angle DBC+\angle ABC)=180^\circ-90^\circ=90^\circ,$$

so $IC=40$. Then the radius of ω is just the distance from I to BC, which is $30\cdot 40/50=\boxed{24}$.

10. Each square's side length is exactly one-quarter of the one adjacent to it from the previous iteration. If the side length of the base square is 1, what is the total perimeter of the figure as the pattern goes to infinity?

Answer: 12

Proposed by Aaron Zhang

Call the side length of the original squares s. The first iteration produces 4 new squares, replacing a segment of length $\frac{s}{4}$ with 3 sides of $\frac{s}{4}$. The next iteration produces 3 new squares on each previously generated square for 12 total, replacing a previous side length of $\frac{s}{16}$ with 3 sides of $\frac{s}{16}$. This pattern continues and we have the sequence $4s+\frac{4\cdot 2s}{4}+\frac{12\cdot 2s}{16}+\frac{36\cdot 2s}{64}\cdots=4s+\sum_{i=0}^{\infty}\frac{2s\cdot 3^i}{4^i}$ which sums to $12s$, giving a final answer of $\boxed{12}$.