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Algebra

1. Alex is now reading *Fahrenheit 451*, which has 158 pages numbered $1-158$. Alex flips through a section of a book, only reading every other page. If the sum of the page numbers he reads happens to be 451, what page did he start on?

Answer: 31

Proposed by Aaron Hu

The page numbers Alex reads are either all even or all odd. Since the page number total is odd, the numbers must all be odd. Suppose Alex reads pages $2m+1, 2m+3, \ldots, 2n-1$ for positive integers $m+1 < n$. Then he reads $n-m$ pages with an average page number of $m+n$, so $451 = (n-m)(n+m)$. Since $m+n < 451$, we get that $n-m = 11, n+m = 41$, so $2m+1 = (41-11)+1 = \boxed{31}$.

2. Find the number of real solutions x to $x^2 + \lfloor x \rfloor = n$, where n is any integer less than or equal to 2023. Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Answer: 4049

Proposed by Jessica Wan

Note x^2 must be an integer. If $x > 0$, substitute $y = x^2$. The equation becomes $y + \lfloor \sqrt{y} \rfloor = n$, and one finds that y can take any value ranging from 1 to $2023 - 44 = 1979$.

If $x < 0$, again substitute $y = x^2$. The equation becomes $y - \lceil \sqrt{y} \rceil = n$, and one finds that y can take any value from 1 to $2025 + 44 = 2069$ (as $2069 - 46 = 2023$).

Thus, altogether, noting that 0 is also a solution, we have $1 + 1979 + 2069 = \boxed{4049}$ solutions.

3. Let $a \star b$ be equal to one of $a+b$, $a-b$, $a \cdot b$, or a/b, each chosen with equal probability. If the expected value of $((((1 \star 2) \star 3) \cdots \star 9) \star 10)$ can be written as $\frac{P_1 P_2 P_3 P_4}{P_5}$ for prime powers $P_1, P_2, \ldots, P_5$ what is $P_1 + P_2 + \cdots + P_5$?

Answer: 4330

Proposed by Aaron Hu

Note that

$$\mathbb{E}(a \star b) = \frac{1}{4}\left[(a+b) + (a-b) + ab + a/b\right] = \frac{a}{4}\left(2 + b + \frac{1}{b}\right) = \frac{a(b+1)^2}{4b},$$

so the desired expected value is

$$\prod_{k=2}^{10} \frac{(k+1)^2}{4k} = \frac{1}{4^9} \cdot \left(\frac{11!}{2}\right)^2 \cdot \frac{1}{10!} = \frac{10! \cdot 11^2}{2^{20}} = \frac{3^4 \cdot 5^2 \cdot 7^1 \cdot 11^2}{2^{12}}.$$

Then the answer is $81 + 25 + 7 + 121 + 4096 = \boxed{4330}$.

4. Let $f(x) = 2x^3 - 21x^2 + 65x - 54$. Given that the real solution to $f(f(f(x))) = 8x$ can be expressed as $\frac{m}{n}$ for relatively prime integers m, n, compute $m^2 + n^2$.

Answer: 13

Proposed by Aaron Hu

Since the equation is $f^3(x) = 2^3x$, ideally we want to be able to solve this by solving something like $f(x) = 2x$. However, this would also require $f(2x) = 4x$ and $f(4x) = 8x$. This happens to be the case. Consider the polynomial $g(x) = f(x) - 2x$. This factors as $g(x) = (2x-3)(x-3)(x-6)$, so for $x = \frac{3}{2}, 2 \cdot \frac{3}{2}, 4 \cdot \frac{3}{2}$, we have $f(x) = 2x$. Then $\frac{3}{2}$ is the desired solution, yielding $3^2 + 2^2 = \boxed{13}$.

Combinatorics

1. Tiger has four $4-$sided dice, each with faces labeled $1-4$, and Andrew has four $6-$sided dice, each with faces labeled $1-6$. They play a game where they both roll all their dice, and whoever has the (single) die with the highest outcome wins. If there's a tie, they flip a fair coin to determine the winner. If the probability that Andrew wins can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m+n$?

Answer: 154

Proposed by Aaron Hu

If all of Andrew's rolls are from $1-4$, then he has a $\frac{1}{2}$ chance of winning. Otherwise, he is guaranteed to win. Then the probability he wins is

$$\left(\frac{2}{3}\right)^4 \cdot \frac{1}{2} + \left[1 - \left(\frac{2}{3}\right)^4\right] = \frac{8}{81} + \frac{65}{81} = \frac{73}{81},$$

so the answer is $73 + 81 = \boxed{154}$.

2. How many five-digit positive integers exist such that its nonzero digits are strictly decreasing from left to right?

Answer: 1287

Proposed by Aaron Hu

If such a number has k nonzero digits, there are $\binom{9}{k}$ ways to choose what these digits are, and then $\binom{4}{k-1}$ ways to choose the placement of these nonzero digits (the leading digit must be nonzero). Summing over $k = 1, \ldots, 5$ and using Vandermonde's identity gives

$$\sum_{k=1}^{5} \binom{9}{k}\binom{4}{k-1} = \sum_{k=1}^{5} \binom{9}{k}\binom{4}{5-k} = \binom{13}{5} = \boxed{1287},$$

as desired.

3. Some cells of a 71×71 grid of squares are colored blue so that any blue cell shares at least one vertex with at most one other blue cell. Find the greatest possible number of blue cells.

Answer: 1728

Proposed by Jessica Wan

For an upper bound, consider the pairs (v, c) consisting of a vertex v and an adjacent blue cell c. Given i blue squares, the number of such pairs is $4i$. The number of vertices adjacent to 2 blue cells is at most i, as 2 neighboring cells share at most 2 vertices and each cell has at most one neighbor. Every other vertex is adjacent to at most 1 blue cell.

Thus, $4i \leq 72^2 + i \implies i \leq 24 \cdot 72 = \boxed{1728}$. This is achievable by coloring all cells that are in rows $1, 2 \pmod 3$ and columns $1 \pmod 2$, where rows and columns are numbered 1 through 71.

4. A row of 2023 buckets are all empty except for the middle bucket. In one move, one can choose any bucket except the first and last and transfer $\frac{2}{3}$ of its contents to its left neighbor and $\frac{1}{3}$ of its contents to its right neighbor. If N is the minimum integer number of gallons of water that one must start with to move at least 2023 gallons of water into the rightmost bucket, find the last three digits of N.

Answer: 128

Proposed by Jessica Wan

We define an invariant on the distribution of the water in the buckets. Give the i^{th} bucket from the left a weight of 2^{i-1}. Note that as $2 = \frac{2\cdot 2\cdot 1}{2(3)} + \frac{2\cdot 1\cdot 2}{1(3)}$, the sum of the weights times the water held for each bucket is an invariant.

This value starts at $N \cdot 2^{1011}$. If the ending configuration has at least 2023 gallons of water in the rightmost bucket, the value is at least $2023 \cdot 2^{2022} + (N - 2023) \cdot 2^0$. Thus, we have $N \cdot 2^{1011} \geq 2023 \cdot 2^{2022} + (N - 2023) \implies N \geq \frac{2023 2^{2022} - 2023}{2^{1011}-1} = 2023\left(2^{1011}+1\right)$.

(Note that it is impossible to move all liquid into the first and last buckets, so our inequality is actually strict, though to calculate $\log_2 N$ this is irrelevant.) To prove that any N satisfying the above inequality is sufficient, consider infinitely operating on the buckets in the order $2, 3, \cdots, 2022, 2, 3, \cdots$. The amount of water not in the first or last bucket decreases by at least $\frac{1}{3^{2022}}$ of its original value after each set of 2021 moves. Thus, as the number of moves approaches infinity, this amount will converge to 0, and we will have at least 2023 gallons of water in the rightmost bucket, as desired.

The minimum integer value of N is then $2023\left(2^{1011}+1\right)+1$, which is equivalent to $2024 \equiv 0 \pmod 8$ and $23\left(2^{11}+1\right)+1 \equiv 3 \pmod{125}$. Thus, the desired answer is $\boxed{128}$.

Geometry

1. Let $ABCD$ be an isosceles trapezoid with $\overline{AB} \parallel \overline{CD}$ and circumcenter O. Given that $\angle AOB = 42^\circ$ and $\angle COD = 138^\circ$, what is minimum possible value of $\angle ABC$ in degrees?

Answer: 114

Proposed by Aaron Hu

The minimum occurs when O is located between $\overline{AB}$ and $\overline{CD}$, which we will assume for the rest of the solution. Since $OA = OB$, we have $\angle ABO = \frac{1}{2}(180^\circ - 42^\circ) = 69^\circ$. From symmetry,

$$2\angle BOC = \angle BOC + \angle DOA = 360^\circ - (\angle ABO + \angle COD) = 180^\circ,$$

so $\angle BOC = 90^\circ$. Then $\angle OBC = 45^\circ$, and the answer is $\angle ABO + \angle OBC = 69 + 45 = \boxed{114}$ degrees.

2. Let $ABCD$ be a parallelogram with $AB = 5$, $BC = 8$, and $CA = 7$. Let O denote the circumcenter of $\triangle ABC$. If OD^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m + n$?

Answer: 172

Proposed by Aaron Hu

From Law of Cosines, we can see that $\angle ABC = \angle CDA = 60^\circ$. Then

$$\angle AOC = 2\angle ABC = 120^\circ = 180^\circ - \angle CDA,$$

so $CDAO$ is cyclic. From Law of Sines, $OA = OC = \frac{AC}{2\sin\angle ABC} = \frac{7}{2\cdot\frac{\sqrt{3}}{2}} = \frac{7}{\sqrt{3}}$, so from Ptolemy,

$$\frac{7}{\sqrt{3}}(8+5) = OC \cdot DA + OA \cdot CD = AC \cdot OD = 7OD,$$

so $OD^2 = \left(\frac{13}{\sqrt{3}}\right)^2 = \frac{169}{3}$, and the answer is $169 + 3 = \boxed{172}$.

3. Two clocks, C_1, C_2, with radii 2023 touch at point P at points 3 and 9 o'clock on the circles, respectively. An ant at P walks counterclockwise along C_1 at the same pace as another ant walking clockwise starting from the 1 o'clock position on C_2. As they walk indefinitely, the midpoint of their positions traces a finite locus. Find the length of this locus.

Answer: 4046

Proposed by Jessica Wan

Translating circle C_2 left by a diameter's length would translate each point of the desired locus left by 2023, not affecting its length. Let Q denote the position of the ant on C_2, and say Q' is the image of Q on circle C_1. Note that the two ants' new positions are always symmetric across a diameter of C_1, connecting the 2 and 8 o'clock positions. One can see that this segment is the exact (translated) locus, and has length $\boxed{4046}$.

4. Consider acute $\triangle ABC$ with orthocenter H. Suppose $\odot(BHC)$ intersects AB, AC a second time at D, E, respectively. If $AH = 13$ and $DE = 24$, then BC can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m + n$.

Answer: 161

Proposed by Aaron Hu

Using directed angles, we have

$$\measuredangle AEB = \measuredangle CEB = \measuredangle CHB = \measuredangle BAC = \measuredangle BAE,$$

implying that $\triangle ABE$ is isosceles. Similarly, $\triangle ACD$ is isosceles. Since H lies on the B and C altitudes in $\triangle ABE, \triangle ACD$, respectively, we have $HE = HA = HD$, so H is the circumcenter of $\triangle ADE$. From the Extended Law of Sines,

$$\sin\angle A = \frac{DE}{2AH} = \frac{24}{2\cdot 13} = \frac{12}{13} \implies \cos\angle A = \frac{5}{13}.$$

Then $AE = \frac{10}{13}AB$, $AD = \frac{10}{13}AC$, so $\triangle AED \sim \triangle ABC$ with similarity ratio $\frac{10}{13}$. We have

$$BC = \frac{13}{10}DE = \frac{13}{10}\cdot 24 = \frac{156}{5},$$

and the answer is $156 + 5 = \boxed{161}$.

Number Theory

1. Consider the $2n$-digit integer $K = 202323\ldots23$, where the digits repeat with 23s after the initial 20. Find the least positive integer n such that K is divisible by 99.

Answer: 69

Proposed by Alex Li

Note that $23 \equiv 5 \pmod 9$ and $23 \equiv 1 \pmod{11}$. Furthermore, $20 \equiv 2 \pmod 9$ and $20 \equiv -2 \pmod{11}$. Therefore, we see that, if there are x occurrences of 23, then

$$2 + 5x \equiv 0 \pmod 9, \quad -2 + x \equiv 0 \pmod{11}.$$

This is equivalent to $x \equiv 5 \pmod 9$ and $x \equiv 2 \pmod{11}$. This gives $x \equiv 68 \pmod{99}$, so the answer is $68 + 1 = \boxed{69}$.

2. How many integers $1 < n \leq 10000$ are such that $p^2 + 2 \mid n$, where p is the smallest prime factor of n?

Answer: 1818

Proposed by Aaron Hu

For $p = 2$, we only require $6 \mid n$, yielding 1666 solutions. For $p = 3$, we have $p^2 + 2 = 11$, so we need to count the odd multiples of 33. There are $9999/33 = 303$ multiples of 33 in the given range, yielding $(303 + 1)/2 = 152$ solutions.

Now, consider when $p > 3$. Then $3 \nmid p$, so $p^2 \equiv 1 \pmod 3$, implying that $3 \mid p^2 + 2 \mid n$. However, by assumption, p is the smallest prime dividing n, so we have a contradiction. Then the answer is $1666 + 152 = \boxed{1818}$.

3. Compute the sum of the distinct prime factors of $12^8 - 1$.

Answer: 380

Proposed by Aaron Hu

We can factor

$$12^8 - 1 = (12^4 + 1)(12^2 + 1)(12 + 1)(12 - 1) = (12^4 + 1) \cdot 145 \cdot 13 \cdot 11.$$

Note that $145 = 5 \cdot 29$, so we currently have prime factors $5, 11, 13, 29$. To factor $12^4 + 1$, note that

$$12^4 + 1 = 144^2 + 1 = F_{12}^2 + 1 = F_{11} \cdot F_{13} = 89 \cdot 233,$$

where F_n is the nth Fibonacci number. We can check that both 89 and 233 are prime, so the final answer is $5 + 11 + 13 + 29 + 89 + 233 = \boxed{380}$.

4. Sequences a_n, b_n with $a_1 = b_1 = 1$ satisfy the recurrences $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$. Find the remainder when $a_{306} \cdot b_{306}$ is divided by 17.

Answer: 6

Proposed by Jessica Wan

First note that a_n, b_n satisfy $a_{n+1} + b_{n+1}\sqrt{2} = (a_n + b_n\sqrt{2})(1 + \sqrt{2})$. Thus, $a_n + b_n\sqrt{2} = (1 + \sqrt{2})^n$. Expanding using the binomial theorem we find that

$$(1 + \sqrt{2})^{17} \equiv 1 + \sqrt{2}^{17} \equiv 1 + 2^8\sqrt{2} \equiv 1 + \sqrt{2} \pmod{17},$$

as $17 \mid \binom{17}{i}$ for $i = 1, 2, \ldots, 16$. Further, $(1 + \sqrt{2})^{289} \equiv 1 + \sqrt{2} \pmod{17}$. Since $306 = 17 \cdot 18$, we have

$$a_{306} + b_{306}\sqrt{2} \equiv (1 + \sqrt{2})^{289} \cdot (1 + \sqrt{2})^{17} \equiv (1 + \sqrt{2})^2 \equiv 3 + 2\sqrt{2} \pmod{17},$$

and the final answer is $3 \cdot 2 = \boxed{6}$.