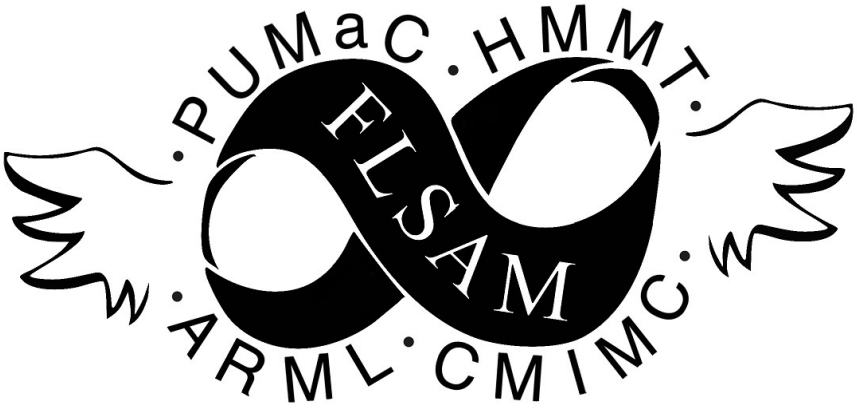
Florida Student Association of Mathematics



2023 Spooky Halloween Contest Individual Round Solutions

October 2023

1. A sorcerer can allocate 15 mana between the range, r, and strength, s, of a spell. If the overall effect is $rs + 2r + 3s$, find the maximum possible effect.

Answer: 94

Proposed by Darsh Patel

We begin by rewriting this as $(r+3)(s+2) - 6$. Now, by the AM-GM inequality,

$$(r+3)(s+2) \leq \left(\frac{r+3+s+2}{2}\right)^2 = \left(\frac{20}{2}\right)^2 = \frac{400}{4}$$

The largest possible value is then $100 - 6 = \boxed{94}$.

Alternatively, using $s = 15 - r$, we have

$$(r+3)(17-r) - 6 = -r^2 + 14r + 45 = -(r-7)^2 + 94,$$

which is maximized at $r = 7$, so the answer is 94.

2. Andrew is addicted to eating candy corn. His left hand can hold 11 candy corn, his right hand can hold 13 candy corn. Every minute, he eats all the candy corn one of his hands can hold, alternating between his left and right hands and starting with his left. What is the smallest perfect square amount of candy corn he could have eaten after any positive integer number of minutes?

Answer: 144

Proposed by Aaron Hu

After an odd number of minutes, Andrew will have eaten $11 \pmod{24}$ candy corn, which is $2 \pmod{3}$ and cannot be a perfect square. After $2n$ minutes for positive integer n, Andrew will have eaten $24n$ candy corn, which equals $\boxed{144}$ for minimal $n = 6$.

3. Yuhan is trick-or-treating. Each house has 50% chance of tricking him and 50% chance of giving him a treat. When he is tricked, he loses all the treats he has gathered thus far and begins a new streak. If the probability Yuhan gathers four treats in one streak before being tricked three times is $\frac{p}{q}$ for relatively prime positive integers p, q, what is $p + q$?

Answer: 4817

Proposed by Yuhan Niu

The important thing to realize is that the streak ending condition can be ignored. The 3 rounds are still independent even though the round ends due to a conditional. Within each round, the probability of not achieving a streak of 4 or greater is $\frac{15}{16}$. The probability that at least one of the rounds has a streak of 4 or greater is $1 - (\frac{15}{16})^3 = \frac{721}{4096}$. The answer is then $\boxed{4817}$.

4. A blueprint for a haunted house is arranged in a 2×2 grid. In each room a silly architect writes a positive integer number of spooky, scary skeletons. On both sides of each row, and on the top and bottom of each column, he writes the product of the two numbers in the row or column. The sum of all twelve written numbers is 2023. What is the smallest possible number of skeletons in the grid?

Answer: 63

Proposed by Yuhan Niu

Let the four numbers be a, b, c and d. The sum of the 12 numbers can be written as $a + b + c + d + 2ab + 2cd + 2ad + 2bc = 2023$. This can be rearranged to $(a + c) + (b + d) + 2(a + c)(b + d) = 2023$.

Dividing by 2 and using SFFT we can get the result $(a+c+\frac{1}{2})(b+d+\frac{1}{2}) = \frac{4047}{4}$. Since a, b, c and d are all integers, the two factors need to have denominators of 2. Now, we need to find the closest two factors, which turn out to be $\frac{57}{2}$ and $\frac{71}{2}$. Therefore, the sum of the four numbers is $a+c+b+d = 28+35 = \boxed{63}$.

5. In triangle AEI, points V, L are constructed on line EI such that $IV = IL = EI/2$, arranged in order E, V, I, L. Further, $S \neq A$ is constructed on AE with $ES = EA$. If $AV = EI = 8$ and $AE = 9$, find SL^2.

Answer: 324

Proposed by Jessica Wan

Construct additionally D on $\overline{EVIL}$ with $DE = EV = 4$. Note $ADSV$ is a parallelogram. Also, $VD = VA = VL$, so $\angle DAL = 90^\circ$. Thus, $SV \perp AL$ and, as $VA = VL$, $SA = SL$. So, $SL^2 = SA^2 = 4AE^2 = \boxed{324}$. (Alternatively, use law of cosines or Stewart's twice.)

6. A ghost has a string "OBOBO...OBOBO" which includes 17 O's and 16 B's. Having discovered a love for the palindromic, find the number of ways it can delete all but 3 characters of the string to leave the word "OBO".

Answer: 816

Proposed by Jayden Lang

Suppose the indices of the 3 letters left are i, j, k (letters are indexed from 1 to 33). Let $i' = \frac{i+1}{2}, j' = \frac{j}{2}, k' = \frac{k-1}{2}$. Our goal is to find the number of triplets of positive integers i', j', k' such $1 \leq i' \leq j' \leq k' \leq 16$. We claim that this is just $\binom{16+2}{3}$. Notice that $\binom{16}{3}$ covers the amount of triplets such that i', j', and k' are all distinct. If we add a "wild card" such that if the numbers p and q are selected, the "wild card" is the minimum of p and q, the $i' \leq' j < k$ case is covered. Similarly, adding another "wild card" covers the $i < j \leq k$ case. Thus, the $1 \leq i' \leq j' \leq k' \leq 16$ case is simply $\binom{18}{3} = \boxed{816}$.

7. Halt! A sphinx demands you answer the following riddle: A monic quadratic polynomial $P(x)$ has two distinct integer roots, whereas $P(P(x))$ only has one root. If $P(0)$ is a multiple of 2024, what is the smallest possible value of $\frac{P(0)}{2024}$?

Answer: 1936

Proposed by Tom Zhang

Let $P(x) = (x+m)(x+n)$. WLOG, we let $m > n$. Since $P(P(x))$ only has one root, $((x+m)(x+n)+m)((x+m)(x+n)+n) = 0$ only has one root. Since $m > n$, we have to restrict $(x+m)(x+n) = -n$ only has one root. If $(x+m)(x+n)+m$ has roots, then $(x+m)(x+n)+n$ must have roots. Thus, we get $(x+m)(x+n)+n = x^2+(m+n)x+mn+n$ only has one root, $(m+n)^2 = 4(mn+n) \implies m-n = 2\sqrt{n}$. Since m is an integer, n must be a perfect square. WLOG, we let $n = k^2, p(0) = k^2(2k+k^2) = k^3(k+2)$ is a multiple of 2024. $2024 = 2^3 \cdot 11 \cdot 23$

It is easy to inspect $k = 44$ is the smallest k to satisfy the condition (as k must be even). The answer is then $\frac{P(0)}{2024} = \frac{44^3 \cdot 46}{2024} = 44^2 = \boxed{1936}$.

8. Jayden's favorite non-zero integers a, b, and c satisfy the spooky property that

$$\frac{a+b-c}{c} = \frac{b+c-a}{a} = \frac{a+c-b}{b}.$$

Find the sum of all the possible values of $\frac{(a+b)(b+c)(c+a)}{abc}$.

Answer: 7

Proposed by Jayden Lang

Let $\frac{a+b-c}{c} = \frac{b+c-a}{a} = \frac{a+c-b}{b} = n$, then we have $(a+b) = (n+1)c$, $(b+c) = (n+1)a$, $(a+c) = (n+1)b$. Adding up the three equations, we get $2(a+b+c) = (n+1)(a+b+c)$, thus $(1-n)(a+b+c) = 0$.

Case 1: $n = 1$. Then $a+b = 2c$, $b+c = 2a$, $a+c = 2b$. Thus, $\frac{(a+b)(b+c)(c+a)}{abc} = \frac{2c \cdot 2a \cdot 2b}{abc} = 8$.

Case 2: $a+b+c = 0$. Then $a+b = -c$, $b+c = -a$, $a+c = -b$, so $\frac{(a+b)(b+c)(c+a)}{abc} = -\frac{abc}{abc} = -1$.

Thus, the sum of all possible values of $\frac{(a+b)(b+c)(c+a)}{abc}$ is $\boxed{7}$.

9. Let ABC be an equilateral triangle with side length 2023. Points D, E lie on ray AC and F, G lie on ray AB such that $DC = CE = FB = BG$, $AD < AE$, and $AF < AG$. The center of the circumcircle of DAG lies on the circumcircle of EAF. Given that the length of CD can be written as $\frac{m}{n}$ in simplest form, find $m+n$.

Answer: 2026

Proposed by Tom Zhang

Due to symmetry, the center of (EAF) lies on the circumcircle of $\triangle DAG$. Let the centers of $(ADG), (EAF)$ be O_1, O_2 respectively and the intersection of two circles be $K \neq A$. Since the two circles are congruent, we have $\triangle O_1O_2K$ is equilateral. Furthermore, $O_1O_2 = O_1A$, which indicates AO_2KO_1 is a rhombus. Thus, $AO_2 = O_2K = DE, AD + CD = 3CD = AC, CD = \frac{2023}{3}, m+n = \boxed{2026}$.

10. A dragon has stolen most of the combo problems from this test. It left behind a clue that reads: $y^2 + 3x = 9x^3y^4$ for real numbers x and y. If the minimum value of $\frac{1}{2x} + \frac{3}{y^2}$ is M, find M^4 for any hope of the subject reappearing on FLOMO.

Answer: 27

Proposed by Tom Zhang

Note $(\frac{1}{2x} + \frac{3}{y^2})^2 = \frac{1}{4x^2} + \frac{9}{y^4} + \frac{3}{xy^2}$. $x^2 = \frac{y^2+3x}{9xy^4} = \frac{1}{9xy^2} + \frac{1}{3y^4} \cdot \frac{9}{y^4} + \frac{3}{xy^2} = 27(\frac{1}{9xy^2} + \frac{1}{3y^4}) = 27x^2$.
The minimum value of $\frac{1}{4x^2} + 27x^2 = M^2 = 3\sqrt{3}$, so $M^4 = \boxed{27}$.