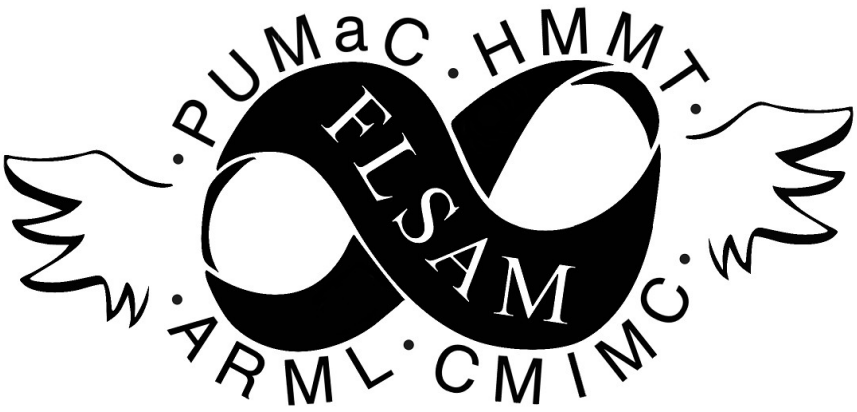
Florida Student Association of Mathematics



2023 Spooky Halloween Contest Relay Solutions

October 2023

Relay Set 1

1. In Transylvania, vampire coffins are given 2-digit numbers with digits in non-decreasing order when read left to right. If no two coffins can have the same number, how many coffins can be built?

Answer: 45

Proposed by Sailalitha Kodukula

Note that the as the first digit must be positive, both digits are from 1 to 9. If there are a_i digits i in the number, $a_1 + a_2 + \cdots + a_9 = 2$, and each 9-tuple $(a_1, \cdots, a_9)$ of non-negative integers determines a unique number. So, the desired answer is $\binom{10}{2} = \boxed{45}$.

2. Let 🙂 be the number you will receive. How many positive integers $n \leq 900$ satisfy:

$$\left\lfloor \frac{n🙂}{11} \right\rfloor = \left\lfloor \frac{(n-1)🙂}{11} \right\rfloor + 5\,?$$

Answer: 81

Proposed by Jessica Wan

Manipulating the equation given 🙂 $= 45$, we find $\left\lfloor \frac{n}{11} \right\rfloor = \left\lfloor \frac{n-1}{11} \right\rfloor + 1$, or $11|n$. Thus, the answer is the number of multiples of 11 at most 900, or $\boxed{81}$.

3. Let 🙂 be the number you will receive. The location of a Halloween festival with rational coordinates (x, y) is encoded in the equation: $xy(xy+4) - 7(x^2+y^2) + 🙂 = 0$. Compute $|x| + |y|$.

Answer: 6

Proposed by Sailalitha Kodukula

Given 🙂 $= 81$, rewrite the equation as $(xy+9)^2 - 7(x+y)^2 = 0$. This step is motivated by the fact that we are told there is one possible value of $|x| + |y|$, so the condition should give multiple restrictions at once. Then, note that two positive, rational squares cannot differ by a factor of 7 as $\sqrt{7}$ is irrational. Thus, $x + y = 0$ and $xy = -9$, implying that $|x| = |y| = 3$, for an answer of $\boxed{6}$.

Relay Set 2

4. Define $\overset{x,y,z}{\text{🎃}} = y + z - 2x$. If b, a, t are the roots of $x^3 - 4x^2 + x + 6$, evaluate

$$\overset{b,a,t}{\text{🎃}} \times \overset{a,t,b}{\text{🎃}} \times \overset{t,b,a}{\text{🎃}}.$$

Answer: 70

Proposed by Darsh Patel

Let $P(x) = x^3 - 4x^2 + x + 6$. By Vieta's, $b + a + t = 4$. Thus, the given expression is equal to

$$(4 - 3b)(4 - 3a)(4 - 3t).$$

This is then equal to

$$27\left(\frac{4}{3} - b\right)\left(\frac{4}{3} - a\right)\left(\frac{4}{3} - t\right) = 27 \cdot P\left(\frac{4}{3}\right).$$

Evaluating this, we get $\boxed{70}$.

Alternatively, one can find the roots of the polynomial: $b = -1$, $a = 2$, $t = 3$. Then, the expression equals $7 \cdot (-1) \cdot (-10) = 70$.

5. Let 💀 be the number you will receive. Consider the infinite sequence $\frac{10💀}{1}, \frac{10💀}{4}, \frac{10💀}{7}, \cdots$ consisting of $10 \cdot 💀$ divided by elements of an arithmetic sequence. Find the number of integer elements of the sequence.

Answer: 10

Proposed by Jessica Wan

Given $💀 = 70$, we seek the number of divisors of 700 that are $1 \pmod 3$. Note $700 = 2^2 \cdot 5^2 \cdot 7$. As $7 \equiv 1 \pmod 3$, we first find the divisors of 100 that work — 1, 4, 10, 25, 100 — and multiply their number by 2, passing on $\boxed{10}$.

6. Let 💀 be the number you will receive. In triangle ABC, T lies on $\overline{CA}$ past A such that $\angle TBA = (6💀)^\circ$ and $\angle TCB = (💀^2 - 4💀)^\circ$. If $BC = 4$ and $CA = 2$, find $[BAT]^2$.

Answer: 108

Proposed by Jessica Wan

First, given $💀 = 10$, note $\angle TBA = \angle TCB = 60^\circ$. Thus, $\triangle TBA \sim \triangle TCB$. Also, $\triangle BCA$ is a right triangle, and $BA = 2\sqrt{3}$. Then, $AT = 6$ and $[BAT]^2 = (6\sqrt{3})^2 = \boxed{108}$.