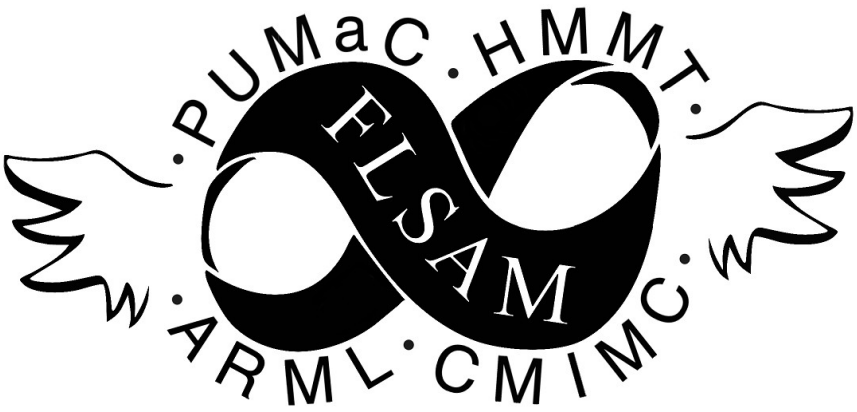
Florida Student Association of Mathematics



2023 Spooky Halloween Contest Team Round Solutions

October 2023

1. On each of eight tables, there sit eight pies of different flavors. There are eight pies of each flavor in the room. Each has eight slices. When each partygoer comes in, they randomly choose a slice of pie to eat. The probability the last two slices of pie eaten are of the same flavor, from the same table, or both can be expressed in simplest form as $\frac{p}{q}$. Find $p+q$.

Answer: 90

Proposed by Jessica Wan

Note that each pair of slices has the same probability of being the last two eaten. Fix the last slice. Then, there are $(8 \cdot 8 - 1) + (8 \cdot 8 - 1) - (8 - 1) = 119$ slices for the second last that satisfy the condition, out of 511 total.

Thus, the probability is $119/511 = 17/73$, for an answer of $\boxed{90}$.

2. Points B, O, N, E are chosen independently at random on the circumference of a circle. If the probability segments BO and NE intersect equals $\frac{a}{b}$ with $\gcd(a,b) = 1$, find $a+b$.

Answer: 4

Proposed by Aaryan Vaishya

Given 4 arbitrarily chosen points, $\frac{1}{3}$ of the labelings of B, O, N, E will have BO intersecting NE. Thus, the desired answer is $\boxed{4}$.

3. A reveler in Candyland starts in the coordinate plane at $(0,7)$ hoping to attend a party at $(28,0)$. However, they must first visit any point on $y = 1$ to withdraw chocolate coins from an ATM and then travel to $y = 8$ to buy a costume. If they take the route of minimal distance, how many units must they travel before they reach the party?

Answer: 35

Proposed by Kevin Liu

Note that the minimal distance must use straight paths going from $(0,7)$ to $(a,1)$ to $(b,8)$ to $(28,0)$. The distance of traveling like so equals the distance of traveling from $(0,21)$ to $(a,15)$ to $(b,8)$ to $(28,0)$, clearly minimized when the points are collinear. Then, the distance is at least $\sqrt{21^2 + 28^2} = \boxed{35}$.

4. Let triangle ABC have $AB = 13$, $BC = 15$, $AC = 14$. Denote the points where angle bisectors AD and BE intersect incircle ω that are closer to the vertices by S and T, respectively, with I the incenter. The area of $\triangle SIT$ can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are integers, $\gcd(a,c) = 1$, and b is not divisible by the square of any prime. Find $a+b+c$.

Answer: 26

Proposed by Aaron Zhang

Let $\overline{AS} = x$ and L be the foot of the perpendicular from I to AC. We have $\overline{AL} = \frac{13+14-15}{2} = 6$. Note that the in-radius r can be found using area formulas: $r \cdot \frac{13+14+15}{2} = 84 \implies r = 4$.

Now, $\overline{AI} = \sqrt{6^2 + 4^2} = 2\sqrt{13}$ and similarly $\overline{BI} = \sqrt{65}$. Applying Law of Cosines to $\triangle AIB$, we now find $\cos AIB = \frac{-1}{\sqrt{5}}$ and thus $\sin AIB = \frac{2}{\sqrt{5}}$. The area of $\triangle SIT$ equals $\frac{1}{2}(4)(4) \cdot \frac{2}{\sqrt{5}} = \frac{16\sqrt{5}}{5}$, so our final answer is $16 + 5 + 5 = \boxed{26}$.

Alternatively, note that $[AIB] = \frac{1}{2} \cdot 4 \cdot 13 = 26$. Then, $[SIT]/[AIB] = 26(SI \cdot IT)/(AI \cdot IB) = 26(4 \cdot 4)/(2\sqrt{13} \cdot \sqrt{65}) = 16/\sqrt{5}$.

5. How many ordered pairs of integers (x, y) where $1 \leq x, y \leq 7$ satisfy $7 \mid x^3 + xy + y^3 + 1$?

Answer: 18

Proposed by Jayden Lang

Rewrite this as $x^3 + xy + y^3 \equiv -1 \pmod 7$. Notice that the only possible residues of a cube modulo 7 are $-1, 0$, and 1. Thus, we can split this problem into cases.

If $x^3 \equiv 0 \pmod 7$, $y^3 \equiv -1 \pmod 7$, yielding the solutions $(7,6)$, $(7,5)$, $(7,3)$. Similarly, $(6,7)$, $(5,7)$, $(3,7)$ work if $y^3 \equiv 0 \pmod 7$.

If $x^3 \equiv y^3 \equiv -1 \pmod 7$, $xy \equiv 1 \pmod 7$, yielding the solutions $(3,5)$, $(5,3)$, $(6,6)$.

If $x^3 \equiv y^3 \equiv 1 \pmod 7$, $xy \equiv 4 \pmod 7$, yielding the solutions $(1,4)$, $(4,1)$, and $(2,2)$.

If one of x^3 and y^3 is $-1 \pmod 7$ and the other $1 \pmod 7$, then $xy \equiv -1 \pmod 7$, yielding the solutions $(2,3)$, $(3,2)$, $(1,6)$, $(6,1)$, $(4,5)$, and $(5,4)$.

In total, there are $\boxed{18}$ solutions.

6. Let $MEOW$ be a parallelogram with point P in its interior such that $MP = 3$, $EP = 9$, $OP = 8$, $WP = 4$, and $\angle MPW + \angle EPO = 180^\circ$. If the area of $MEOW$ can be written in simplest terms as $a\sqrt{b}$, find $a + b$.

Answer: 30

Proposed by Andrew Xing

Because of our angle condition, we are inspired to translate the parallelogram so that OW is put onto ME, creating a new cyclic quadrilateral $P'MPE$. The area of this quadrilateral is $12\sqrt{6}$ by Brahmagupta's. This is equivalent to the area of MPE and WPO combined. Similarly, the sum of the areas of MPW and EPO is $12\sqrt{6}$. This leads us to a final area of $24\sqrt{6}$ and an answer of $\boxed{30}$.

7. Positive integers a and b are such that $\log_{\gcd(a,b)} (\operatorname{lcm}(a,b))$ is an integer greater than 2. Given that $a + b = 35 \gcd(a,b)$, find $\operatorname{lcm}(a,b)$.

Answer: 1296

Proposed by Darsh Patel

Let n be such that $\operatorname{lcm}(a,b) = \gcd(a,b)^n$. Then for any prime p, we have

$$\max\{\nu_p(a), \nu_p(b)\} = n \min\{\nu_p(a), \nu_p(b)\}$$

Taking x to have the same power of p as a when $\nu_p(a) < \nu_p(b)$ and likewise for y, we have that

$$a = xy^n, b = x^n y$$

and $\gcd(a,b) = xy$ and $\operatorname{lcm}(a,b) = x^n y^n$. So, $a + b = 35 \gcd(a,b)$ is equivalent to

$$x^{n-1} + y^{n-1} = 35$$

which has its only solution at $\{x, y\} = \{2, 3\}$ and $n = 4$. Then $\operatorname{lcm}(a,b) = 2^4 \cdot 3^4 = \boxed{1296}$.

8. (Andrew Xing) A lonely witch begins a ritual on Halloween. During the first minute, she summons one toad. Each next minute, the toads summoned in the last minute each summon three more. Find the number of divisors of the smallest n such that after n minutes the number of toads is divisible by $2^2 \cdot 5^5 \cdot 13^3$.

Answer: 90

Proposed by Andrew Xing

After n minutes, there are $1 + 3 + \cdots + 3^{n-1} = (3^n - 1)/2$ toads. We notice that n must be divisible by 2 for our first condition, that $8|3^n - 1$. From here, we can use LTE to find the values of n such that $3^n - 1$ is divisible by 5^5 and 13^3.

For 5^5, we find $n = 4k$ with $5^4|k$. Similarly, for 13^3, we find $n = 3j$ with $13^2|j$. Thus, the smallest n is $2^2 \cdot 3 \cdot 5^4 \cdot 13^2$, and it's number of divisors is $3 \cdot 2 \cdot 5 \cdot 3 = \boxed{90}$.

9. Tom is trapped in a haunted house with doors labelled 1 to 999. A message on the wall tells him that the square of the exit's label ends with three identical digits. Compute the sum of all door numbers that could be the exit.

Answer: 6500

Proposed by Tom Zhang

Multiples of 100 give 000 and thus satisfy the condition. In this case, the sum of such x will be 4500.

Since perfect squares can only end with $1, 4, 5, 6, 9$ and perfect squares will leave a remainder of either 1 or 0 when divided by 4, the only other possible ending will be 444. There are two cases:

Case 1: $x = 10a + 2$

$(10a + 2)^2 = 1000b + 444$, $b = \frac{5a^2+2a-22}{50}$, $5a^2 + 2a - 22 \equiv 0 \pmod{50}$. Thus, a is a multiple of 2 with a remainder of 1 $\pmod 5$, so $a \equiv 6 \pmod{10}$. Let $a = 10k + 6$.

$5a^2 + 2a \equiv 20k + 192 \equiv 22 \pmod{25} \implies k = 4, 9$. As a result $a = 46, 96, x = 462, 962$.

Case 2: $x = 10a + 8$

$(10a + 8)^2 = 1000b + 444$, $5a^2 + 8a \equiv 19 \pmod{50}$. Thus, $a \equiv 1 \pmod 2$, $a \equiv 3 \pmod 5 \implies a \equiv 3 \pmod{10}$. Let $a = 10k + 3$.

$5a^2 + 8a \equiv 80k + 19 \equiv 19 \pmod{50} \implies 5|k$. The solutions in this case are $x = 38, 538$.

The desired answer is then $538 + 38 + 462 + 962 + 4500 = \boxed{6500}$.

10. (Aaron Zhang) A magician has drawn a 5×5 square array of 25 evenly spaced dots. Confused by the ritualistic pentagram, he decides on a less involved shape. How many distinct non-degenerate quadrilaterals can he draw with vertices chosen from the array? (A quadrilateral is non-degenerate if no three vertices are collinear and it does not cross itself.)

Answer: 9498

Proposed by Aaron Zhang

We will use complementary counting, discarding degenerate quadrilaterals. We have $\binom{25}{4}$ total ways to choose vertices. The only degenerate quadrilaterals are when at least three vertices are collinear. If all 4 points are collinear, there are 10 lines parallel to the x and y axis and 2 diagonals which each have 5 points, for $12\binom{5}{4}$ cases and 4 lines of 4 points parallel to the diagonals for $4\binom{4}{4}$ cases.

The same goes for when 3 points are collinear, but the number of points where the 4th vertex can go is determined by the line the other 3 lie on. For the 12 5-point lines, we have 20 other positions for $12\binom{5}{3} \cdot 20$, and the 4 4-point lines have 21 other positions for $4\binom{4}{3} \cdot 21$. The three-point lines can either form a $2 - 2 - 2\sqrt{2}$ right triangle, of which there are 4, or a $2 - 4 - 2\sqrt{5}$ right triangle, of which there are 12. Each one allows 22 options for the 4th point for a total of $16\binom{3}{3} \cdot 22$ cases. Subtracting all the extraneous cases, we have a final answer of $\boxed{9498}$ quadrilaterals.