Florida Student Association of Mathematics

2022 Winter Florida Online Math Open

December 26, 2022 - January 1, 2023

Solutions

1. A toddler's toy set includes three blocks in the shape of a rectangular prism, a triangular prism, and a cylinder. A palette contains holes with the shapes of a rectangle, triangle, and circle. The rectangle can fit any of the three holes, the triangle can fit only the triangular prism hole, and the circle can fit only the cylinder hole. How many ways are there to assign distinct blocks to each hole so that at least two blocks fit their hole?

Answer: 3

Proposed by Yuhan Niu

Split into cases, where each one differs in which block is matched with the rectangle hole. If the rectangular prism is matched with the rectangle hole, there is only one arrangement of the other two blocks that is valid. If the triangular prism is matched with the rectangle hole, one arrangement is valid. If the cylinder is matched with the rectangle hole, one arrangement again is valid. The three cases each have one valid arrangement, resulting in $\boxed{3}$ arrangements total.

2. Yuhan chooses a random positive integer divisor d of 8658. Let p be the probability that d is a palindrome with more than one digit. Compute $420p$.

Answer: 70

Proposed by Yuhan Niu

Prime factorize 8658 into $2 \cdot 3^2 \cdot 13 \cdot 37$. From here, realize that there are 4 non-single digit palindromes (111, 222, 333, and 666), and 24 factors total. This is a probability of $\frac{1}{6}$. $420 \cdot \frac{1}{6} = \boxed{70}$.

3. Compute the number of terms that have rational coefficients in the expansion of $(x\sqrt[8]{4} + y\sqrt[4]{8})^{1000}$ when like terms are combined.

Answer: 501

Proposed by Aaron Hu

Note that $\sqrt[8]{4} = 2^{1/4}$ and $\sqrt[4]{8} = 2^{3/4}$, so

$$\begin{aligned}(x\sqrt[8]{4} + y\sqrt[4]{8})^{1000} &= \sum_{k=0}^{1000} 2^{k/4+3(1000-k)/4} x^k y^{1000-k} \\ &= \sum_{k=0}^{1000} 2^{750-k/2} x^k y^{1000-k}.\end{aligned}$$

Then k must be even so the answer is $\boxed{501}$.

4. Consider triangle ABC. Let the circle with diameter BC intersect AB at D and AC at E. Let BE and CD intersect at X. Let line AX intersect BC at F. If $BF = 2$, $CF = 7$, and $AF = 7$, what is the length of XF?

Answer: 2

Proposed by Xuzhou Ren

Let line AX intersect the circumcircle of ABC at X'. Observe that $\angle BDC = 90 = \angle BEC$, so X is actually the orthocenter. Also, note that by power of a point $X'F = 2$. Since X is the reflection of X' across F, $XF = X'F = \boxed{2}$.

5. A knight at point (x, y) in the coordinate plane can move to either $(x+1, y+2)$ or $(x+2, y+1)$. How many ways can a knight move from $(1, 0)$ to $(20, 23)$?

Answer: 2002

Proposed by Aaron Hu

We can solve the system $a + 2b = 19, 2a + b = 23$ to get that the knight must move $(x, y) \to (x+1, y+2)$ 9 times and $(x, y) \to (x+2, y+1)$ 5 times, for a total of $\binom{14}{5} = \boxed{2002}$ ways.

6. Triangle ABC has sides of length 49, 50, and 51. Points D and E are chosen on distinct sides of ABC such that $\overline{DE}$ bisects the area of the triangle. What is the minimum possible integer value of the sum of the distances from D and E to the common vertex of their edges?

Answer: 70

Proposed by Aaron Hu

Suppose the common vertex is A. From AM-GM,

$$AD + AE \geq 2\sqrt{AD \cdot AE} = \sqrt{2 \cdot AB \cdot AC}.$$

The RHS is minimized when $\{AB, AC\} = \{49, 50\}$, yielding

$$AD + AE \geq \sqrt{2 \cdot 49 \cdot 50} = \boxed{70},$$

which is attained when $AD = AE = 35$.

7. Aaron is taking a 10-question multiple-choice test. For each question, he has answer choices A, B, C, and D. Aaron hates a key when its answer choice switches. For example, he hates the key of answer choices $AAABBBCCCC$ 2 times and the key $AAAAAAAAAA$ 0 times. If the expected number of times Aaron will hate a test's key is $\frac{m}{n}$, where m, n are relatively prime positive integers, find $100m + n$.

Answer: 2704

Proposed by Andrew Xing

For each answer choice past the first, there is a $\frac{3}{4}$ chance that it is different from the one before it. Thus, our expected value is equal to $\frac{3}{4} \cdot 9 = \frac{27}{4}$ and our answer is $\boxed{2704}$.

8. Jessica has four sticks with lengths 2021, 2022, 2023, and 2026. First, she forms a cyclic quadrilateral with area A using the four sticks. She then joins the two sticks with lengths 2021 and 2023 together to form a longer stick. She forms a triangle with area B using the three sticks she now has. Given that $\frac{A}{B}$ can be expressed as $\frac{m}{n}$ for coprime positive integers m, n, compute $100m + n$.

Answer: 4502

Proposed by Aaron Hu

Note that the quadrilateral has side lengths $x-2, x-1, x, x+3$ and semiperimeter $2x$, where $x = 2023$. From Brahmagupta, the area of the quadrilateral is $\sqrt{(x-3)(x)(x+1)(x+2)}$. The triangle has side lengths $x-1, x+3, 2x-2$ and semiperimeter $2x$, so from Heron, the area of the triangle is $\sqrt{(2x)(2)(x-3)(x+1)}$. Then the ratio of areas is just $\sqrt{x+2}/2 = \sqrt{2025}/2 = 45/2$, so the answer is $\boxed{4502}$.

9. There are two distinct solutions α, β to the equation $4\cos x + 3\sin x + \frac{\sqrt{69-\sqrt{420}}}{2} = 0$ for $0 < x < 2\pi$. If the value of $\cos(\alpha+\beta)$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $100m + n$?

Answer: 725

Proposed by Xuzhou Ren

Plug in α and β to obtain

$$4\cos\alpha + 3\sin\alpha + \frac{\sqrt{69-\sqrt{420}}}{2} = 0$$
$$4\cos\beta + 3\sin\beta + \frac{\sqrt{69-\sqrt{420}}}{2} = 0$$

Subtracting the two equations, we get that $4\cos\alpha - 4\cos\beta + 3\sin\alpha - 3\sin\beta = 0$.

By sum to product, we have $-4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + 3\cos\frac{\alpha+\beta}{2} sin \frac{\alpha-\beta}{2} = 0$.

Observe that $\sin\frac{\alpha-\beta}{2} \neq 0$ because $\alpha \neq \beta$. Therefore, we can divide by $\sin\frac{\alpha-\beta}{2}$ to get that $4\sin\frac{\alpha+\beta}{2} = 3\cos\frac{\alpha+\beta}{2}$. Therefore, $\cos\frac{\alpha+\beta}{2} = \pm\frac{4}{5}$. Using double angle, we get that $\cos(\alpha+\beta) = 2 \cdot \frac{4}{5}^2 - 1 = \frac{7}{25} \longrightarrow \boxed{725}$.

10. Consider the function $f(x) = \lfloor \pi x \rfloor - \lceil ex \rceil$. Let m and n denote the number of positive integer solutions to $f(x) = f(x-1)$ and $f(x) = f(x-1) + 2$, respectively, where $1 \leq x \leq 1000$. Compute $m - n$.

Answer: 578

Proposed by Aaron Hu

Define $g(x) := f(x) - f(x-1)$, and let m, a, n denote the number of solutions to $g(x) = 0, 1, 2$, respectively. Note that

$$\lfloor \pi(x+1) \rfloor - \lfloor \pi x \rfloor \in \{3, 4\}, \quad \lceil e(x+1) \rceil - \lceil ex \rceil \in \{2, 3\},$$

so $g(x) \in \{0, 1, 2\}$ for all x, so $m + a + n = 1000$. Additionally,

$$a + 2n = g(1) + g(2) + \cdots + g(1000) = f(1000) - f(0) = 422,$$

so subtracting the two equations gives $m - n = \boxed{578}$.

11. Call a nonnegative integer *cute* if the sum of its digits is a multiple 7. Tiger lists all cute numbers in order but stops after listing 2023. How many numbers does he list?

Answer: 283

Proposed by Tom Zhang

We divide it into different cases, the sum is $7, 14, 21, 28$. It is obvious that only 1999 satisfies that the sum is 28. For any number between $[2000, 2023]$, only $2005, 2014, 2023$ satisfy the question, so now, we only need to consider integers between $(0, 1999)$ such that the sum of digits is a multiple of 7

1st case:

The sum is 7, $0 < \overline{abc} < 1000$ $a + b + c = 7$ tells that there are $\binom{7+3-1}{3-1} = 36$ cases.

Then, for $999 < \overline{1abc} < 2000$, $a + b + c = 6$ tells that there are $\binom{6+3-1}{3-1} = 28$ cases. Add together, there are 64 different cases

2nd case:

The sum is 14. For $0 < \overline{abc} < 1000$, $a+b+c=14$ tells that there are $\binom{14+3-1}{3-1} = 120$ cases. However, a, b, c can't be greater than 9. Thus, triples $(0,0,14)$, $(0,1,13)$, $(0,2,12)$, $(0,3,11)$, $(0,4,10)$, $(1,1,12)$, $(1,2,11)$, $(1,3,10)$, $(2,2,10)$ have to be removed from the cases. There are $3 \cdot 3 + 6 \cdot 6 = 45$ cases can be formed with those non-working triples, thus $120 - 45 = 75$ cases work.

Then, for $999 < \overline{1abc} < 2000$, $a+b+c=13$, there are 105 possible cases. Similarly, we have to get rid of those triples that can't work. $(0,0,13)$, $(0,1,12)$, $(0,2,11)$, $(0,3,10)$, $(1,1,11)$, $(1,2,10)$ doesn't work. There are $3 \cdot 2 + 6 \cdot 4 = 30$ cases, thus $105 - 30 = 75$ cases work. In all, $75 + 75 = 150$ cases in this situation.

3rd case:

The sum is 21. We process the same procedure as above, but it is easier to find which triples "work" in this case.

$(9,9,3)$, $(9,8,4)$, $(9,7,5)$, $(9,6,6)$, $(8,8,5)$, $(8,7,6)$, $(7,7,7)$ work, there are $3 \cdot 3 + 6 \cdot 3 + 1 = 28$ cases.

Now when the sum is 20, triples $(9,9,2)$, $(9,8,3)$, $(9,7,4)$, $(9,6,5)$, $(8,8,4)$, $(8,7,5)$, $(8,6,6)$, $(7,7,6)$ work, there are $3 \cdot 4 + 6 \cdot 4 = 36$ cases, sum up to get 64 cases.

Of course, 0 works.

Thus, 2023 is the $64 + 150 + 64 + 1 + 3 + 1 = \boxed{283}$rd *cute* number.

12. Consider function $f(x) = x^2 + mx + n$, where m, n are real constants. How many ordered pairs (m, n) are there such that there exist distinct integers a, b satisfying $f(a) = f(b) = 69$ and $f(a+b) = 420$?

Answer: 8

Proposed by Tom Zhang

Let $g(x) = f(x) - 69$, then, we have $g(x) = (x-a)(x-b), f(a+b) = g(a+b) + 69 = ab + 69 = 420, ab = 351$, then, we realize that $351 = 3^3 \cdot 13$, which has $(3+1)(1+1) = 8$ factors. Thus, for m is positive and m is negative, both of them have $\frac{8}{2} = 4$ cases, so there are $\boxed{8}$ certain $f(x)$.

13. If the area of a triangle with side lengths $8, 15x, 17x$ is maximized at $x = \frac{\sqrt{m}}{n}$ for positive integers m, n, and m square-free, find $100m + n$.

Answer: 51408

Proposed by Andrew Xing

We proceed with Heron's formula. The area of the triangle can be represented with

$$\sqrt{(16x+4)(4-x)(4+x)(16x-4)}$$

as our semiperimeter is equivalent to $16x + 4$.
From here, we factor out two 4's and use difference of squares to get $4\sqrt{(16x^2-1)(16-x^2)}$. Expanding gets

$$4\sqrt{-16x^4 + 257x^2 - 16}$$

We can minimize this with $-b/2a$, getting $257/32 = x^2$. Thus, $x = \sqrt{514}/8$ and answer extraction gives $\boxed{51408}$.

14. Consider a segment of length 1. Five points are selected randomly and independently along the length of the segment, partitioning the segment into six sub-segments. The probability that the resulting sub-segments can connect to form a hexagon can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

Answer: 1316

Proposed by Ahan Mishra

In order for the subsegments to form a valid hexagon, the largest subsegment must be less than the sum of the lengths of all other subsegments, so the length of the largest subsegment is less than $1/2$. The largest subsegment can be in six different positions relative to the other subsegments. WLOG, rearrange the subsegments so that the largest is the leftmost. In order to not be a hexagon, all 5 partitions must be on the right half of the segment. The probability of this is $(1/2)^5 = 1/32$, so the desired probability is is $1 - 6(1/32) = 13/16 \to \boxed{1316}$.

15. Consider $x, y, z \in R$ such that $\frac{(x+2)^2}{2} + \frac{(y+9)^2}{9} + \frac{(z+25)^2}{25} = 1$. What is the difference between the maximum and minimum possible values of $x + y + z$?

Answer: 12

Proposed by Xuzhou Ren

We are trying to make a nice trig sub, but we have 3 terms that sum to 1. Therefore, let $\frac{(x+2)^2}{2} = \cos^2\alpha$, $\frac{(y+9)^2}{9} = \sin^2\alpha\cos^2\beta$, $\frac{(z+25)^2}{25} = \sin^2\alpha\sin^2\beta$. This satisfies the condition. Therefore, we have $x = \sqrt{2}\cos\alpha - 2, y = 3\sin\alpha\cos\beta - 9, z = 5\sin\alpha\sin\beta - 25$. From here we can write $x + y + z = \sqrt{2}\cos\alpha + 3\sin\alpha\cos\beta + 5\sin\alpha\sin\beta - 36$. We now use the identity $a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x + \arctan\frac{b}{a})$ to get $\sqrt{2}\cos\alpha + \sin\alpha(\sqrt{34}\sin(\beta + K))$. The K is not relevant for the solution. We apply the same identity again to get $\sqrt{2 + 34\sin^2\gamma}\sin(\alpha + \delta)$. Again, the γ and δ are just variables and are irrelevant. Since the range of sin is $[-1, 1]$, we get that the minimum possible value of this equation is -6 and the max possible is 6 for a difference of $\boxed{12}$.

16. How many ways are there to tile a 10×18 board with L-shaped tetrominoes?

Answer: 0

Proposed by Andrew Xing

We can prove that there are no ways to tile this board with L dominoes using invariants. We can cover the first column with 0's, the second with 1's, and alternate. All configurations of the L domino will sum to an odd number. Because there are an odd number of L's, as $180/4 = 45$, the total sum of the L's will be odd. However, the total sum of the board will be even, and we cannot tile the board. Thus, the answer is $\boxed{0}$.

17. A random point is selected with coordinates (x, y) satisfying $0 \le x, y \le 1$. The probability that point $(4\sqrt{3}x + 6y, 2x + 6\sqrt{3}y)$ lies inside the circle $(x - 8\sqrt{3})^2 + (y - 8)^2 = 12$ can be expressed as $\frac{a\pi - b\sqrt{c}}{d}$ for positive integers a, b, c, d satisfying $\gcd(a, b, d) = 1$ and c square-free. Find $1000a + 100b + 10c + d$.

Answer: 2390

Proposed by Yuhan Niu

The region that point $(4\sqrt{3}x + 6y, 2x + 6\sqrt{3}y)$ can be plotted in is the parallelogram with vertices $(0, 0)$, $(4\sqrt{3}, 2)$, $(6, 6\sqrt{3})$, and $(4\sqrt{3} + 6, 2 + 6\sqrt{3})$. We are looking the area of intersection between this parallelogram and the circle divided by the total area of the parallelogram. Drawing the two shapes, it appears that they intersect along the right side of the parallelogram. Solving for the equation of the line that represents the right side of the parallelogram gets us $y = \sqrt{3}x - 10$. This can be found to intersect the circle at points $(6\sqrt{3}, 8)$ and $(7\sqrt{3}, 11)$. Verify that the top side of the parallelogram doesn't intersect the circle too, which it doesn't. The distance between the two intersection points is $\sqrt{12}$, the same as the radius of the circle. This means the intersecting area is a segment with central angle 60°. We use the lemma that this segment has an area of $r^2(\frac{\pi}{6} - \frac{\sqrt{3}}{4})$, where r is the radius of the circle. The intersection area is $2\pi - 3\sqrt{3}$, while the total area can be found with Shoelace Method to be 60. This results in a probability of $\frac{2\pi - 3\sqrt{3}}{60}$. Therefore the answer is $\boxed{2390}$.

Proof of Lemma: WLOG, let circle O have a radius of r. Draw a chord corresponding to an arc of 60° and label the endpoints A and B. The area of the segment is [sector AOB] $-$ [$\triangle AOB$]. This equals $\frac{\pi r^2}{6} - \frac{r^2\sqrt{3}}{4} = r^2(\frac{\pi}{6} - \frac{\sqrt{3}}{4})$

18. Let S be the set of all positive integers n such that $9^{3n} - 5^{2n}$ is divisible by 2^n. Find the sum of the elements in S.

Answer: 29

Proposed by Andrew Xing

We begin by taking out the 3 and 2 from our exponents to get

$$729^n - 25^n$$

Using LTE for 2 gives, $\nu_2(729^n - 25^n) = \nu_2(704) + \nu_2(754) - 1 + \nu_2(n)$. Thus, we know that $n < 6 + \nu_2(n)$. $n = 1, 2, 3, 4, 5, 6, 8$ work, and our answer is $\boxed{29}$.

19. Call a positive integer n *consistent* if the following is true: there exists a positive integer y such that if x is a positive integer where 2^{n+x} and 5^x have the same left-most digit, then this digit is equal to y. Compute the product of all consistent numbers.

Answer: 48

Proposed by Karthik Vedula

If we multiply two numbers beginning with 1, then their product must begin with 1, 2, or 3. If we multiply two numbers beginning with 2, then their product must begin with $4, 5, 6, 7, 8$. We can continue this logic by saying that if we multiply two numbers beginning with x, then their product must begin with some number in $[x^2, (x+1)^2)$. We focus on these 9 intervals.

Note that the product of 2^{n+x} and 5^x begins with 2^n. If the leading digits of 2^n could be classified as somewhere in the interior of two or more of these nine intervals based on the leading digits, then the common leading digit of 2^{n+x} and 5^x can attain more than one possibility due to the Kronecker's Approximation Theorem. Since the intervals "wrap" around the possibilities of leading digits twice, we must look at the boundary cases due to their extremely constraining nature.

For example, consider $n = 6$. The product in question is of the form $6400\ldots0$. This would normally be a part of the 2-interval and the 8-interval, but in order to be a part of the 8-interval, both 2^{n+x} and 5^x would have to be of the form $800\ldots0$, impossible. This means a number is consistent if and only if 2^n is the square of a one-digit number. This gives $n = 2, 4, 6$, so our answer is $\boxed{48}$.

20. The equation $x^4 = 4x^3 + 6x^2 + 4x + 1$ has two real solutions a, b. There exists positive rational numbers r, s such that $|a| + |b| = 2^r + 2^s$. The value of $r^2 + s^2$ is equal to $\frac{m}{n}$ for coprime, positive integers m, n. Compute $100m + n$.

Answer: 3708

Proposed by Karthik Vedula

The original equation can be manipulated as follows:

$$-x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \implies (x+1)^4 = 2x^4 \implies (x+1)^2 = x^2\sqrt{2} \implies x + 1 = \pm x\sqrt[4]{2}.$$

We now have that the roots are $\frac{1}{\sqrt[4]{2}-1}$ and $\frac{-1}{\sqrt[4]{2}+1}$. The second root is negative, so we have

$$2^r + 2^s = \frac{1}{\sqrt[4]{2}-1} + \frac{1}{\sqrt[4]{2}+1} = \frac{2\sqrt[4]{2}}{\sqrt{2}-1} = 2\sqrt[4]{2}(\sqrt{2}+1) = 2^{7/4} + 2^{5/4}.$$

We now have to compute $\left(\frac{7}{4}\right)^2 + \left(\frac{5}{4}\right)^2 = \frac{37}{8}$, so our answer is $100 \cdot 37 + 8 = \boxed{3708}$.

21. Alex needs help evaluating this series. If the sum

$$1^2 \cdot 2! + 2^2 \cdot 3! + 3^2 \cdot 4! + \ldots + 100^2 \cdot 101!$$

can be represented as $a \cdot b! + 2$ for positive integers a, b with b maximized, find $100a + b$.

Answer: 10002

Proposed by Andrew Xing

We need to use a telescoping series to solve this problem. With enough experimenting, we notice that

$$x^2(x+1)! = (x+2)(x-1)(x+1)! - (x+1)(x-2)(x!)$$

Thus, each term cancels with the one before it except for the last and first, leaving $(100+2)(100-1)(100+1)! - 2(-1)1! = (99)(102)! + 2$. So, $100a + b = \boxed{10002}$.

22. The only real root of $(3x-1)(\sqrt{9x^2-6x+5}+1) + (x-4)(\sqrt{x^2-8x+20}+1) = 0$ can be written in the simplest form of $\frac{m}{n}$, find $100m + n$

Answer: 504

Proposed by Tom Zhang

Call function $f(k) = k(\sqrt{k^2+4}+1)$, then we want to find $f(3x-1) + f(x-4) = 0$. Realize that $f(-k) = -f(k)$, telling $f(k)$ is odd. Then, we just need to assure $3x - 1 = 4 - x$, $x = \frac{5}{4}$ so $100a + b = \boxed{504}$

23. There are 16 pool balls in total: 1 cue ball, 1 8-ball, and 7 pairs of matching colored solid and stripe balls. A game is won if a player sinks the 8-ball. A player can only attempt to sink the 8-ball once he or she sinks all the balls of their pattern, which is chosen at the beginning of the game. A player's turn continues if they sink a ball, and ends if they fail to sink one. Assuming no misplays occur, there are n different orders of balls sunken if there are 5 turns in total and the player who chose stripes wins. Compute the remainder when n is divided by 10^4.

Answer: 7440

Proposed by Yuhan Niu

Due to the conditions, the player who chose stripes must go first. This also means the stripes player must take 3 turns while the solid player take only 2. There are $7! \cdot \binom{7+3-1}{3-1} = 181440$ ways for the 8 balls that the stripes players sinks to be separated into the three turns in such a way that he or she wins. For the solids player, for each m number of balls that is sunk, from 0 to 7, there are

$$\binom{7}{m} m! \binom{m+2-1}{2-1} = \frac{7!(m+1)}{(7-m)!}$$

orderings for which the balls can be sunken in. Sum this from $m = 0$ to 7 results in 95901. These two numbers then need to be multiplied by each other, resulting in 17400277440. The last 4 digits of this number is $\boxed{7440}$.

24. Find the sum of $(r_1 + r_2 - 1)(r_3 \cdot r_4 - 1)$ over all ordered quadruples (r_1, r_2, r_3, r_4) of 4 distinct roots of the quintic polynomial

$$2x^5 - (a+1)x^4 + (2a+2)x^3 - (3a+1)x^2 - a,$$

where a is some constant.

Answer: 96

Proposed by Jessica Wan

Note that the equation becomes $\sum_{\text{cyc}} r_1r_3r_4 + r_2r_3r_4 - r_3r_4 - r_1 - r_2 + 1$ over all permutations of the roots. By Vieta's, $\sum_{\text{cyc}} r_1r_2r_3 = (3a+1)/2$, $\sum_{\text{cyc}} r_1r_2 = (2a+2)/2$, $\sum_{\text{cyc}} r_1 = (a+1)/2$.

Thus, the sum equals $2 \cdot (3a+1)/2 \cdot 3! \cdot 2! - (3a+2)/2 \cdot 2! \cdot 3! - 2 \cdot (a+1)/2 \cdot 4! + 120 = 12(3a+1) - 6(2a+2) - 24(a+1) + 120 = \boxed{96}$.

25. Stephen Curry has a career 3-point percentage of 42.8%. After he shoots a 3-pointer, the ball can roll an integer distance anywhere between 0 to 10 feet (inclusive), with a probability proportional to $10 - d$, where d is the distance rolled. Curry comes into the gym and shoots 10 shots regardless of whether he makes them, but after 10, if he misses one he stops shooting (ignoring the results of the first 10 shots). The expected accumulated distance the ball rolls over all of his shots can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $100m + n$.

Answer: 504143

Proposed by Yuhan Niu

Due to linearity of expectation, we can treat any singular shot to have a constant expected rolled distance, and just multiply this distance by the expected number of shots he takes. To find the expected rolled distance, first the proportion constant k such that $k(10 + 9 + 8 + \ldots + 1 + 0) = 1$ needs to be found. This is $k(55) = 1$, so $k = \dfrac{1}{55}$. This means the resultant probabilities are $\dfrac{10}{55}, \dfrac{9}{55}, \cdots \dfrac{1}{55}, \dfrac{0}{55}$. Using these to find the expected rolled distance results in 3. For the expected number of shots taken, this is the sum of 10 and the value of $\sum_{n=0}^{\infty} n(1-p)p^{n-1}$, where p is the probability Steph makes the shot. Using the lemma $\sum_{n=0}^{\infty} np^n = \dfrac{p}{(1-p)^2}$, we get that the expected number of shots taken is $10 + \dfrac{1}{1-p}$, or $10 + \dfrac{1}{\frac{572}{1000}}$. This simplifies to $\dfrac{1680}{143}$. Multiplying by 3 results in $\dfrac{5040}{143}$, so $100m + n = \boxed{504143}$.

26. Alex likes numbers that can be written as $5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}$, with a_i distinct non-negative integers. Find the sum of all the numbers Alex likes up to 2^{14}.

Answer: 571876

Proposed by Jessica Wan

First, note that $1 + 5 + ... + 5^6 = 19531 > 2^{14}$, and $5^7 > 2^{14}$. It is also clear that any 'good' number has a unique representation in Alex's format.

We first consider the complement: 'good' numbers between 16385 and 19531. As $1 + 5 + ... + 5^5 < 2^{14}$, it is clear all such numbers have 5^6 in their representation. There are two cases, if 5^5 is included or not.

If the representation includes $5^6 + 5^5 = 18750$, then adding any subset of $\{5^4, 5^3, ..., 1\}$ will produce a 'good' number in this range. If the representation does not include 5^5, then one can check only 4 numbers satisfy the conditions: $5^6 + 5^4 + 5^3 + 5^2 + a \cdot 5 + b \cdot 1$, where $a, b \in \{0, 1\}$.

Now, the sum of all 'good' numbers at most 19531 is $(1 + 5 + ... + 5^6) \cdot 2^6 = 19531 \cdot 64 = 1249984$. The sum of those over 16384 is $((5^6 + 5^5) \cdot 2^5 + (1 + 5 + ... + 5^4) \cdot 2^4) + ((5^6 + 5^4 + 5^3 + 5^2) \cdot 4 + (5 + 1) \cdot 2) = 600000 + 12496 + 65600 + 12 = 678108$.

The final sum is then $\boxed{571876}$.

27. Let the roots of the polynomial $x^3 - 3x^2 - 2018x + 2020 = \sqrt{2022}$ be r_1, r_2, r_3. Compute the maximum possible value for $r_1 + r_2r_3$.

Answer: 3

Proposed by Jessica Wan

First, we substitute $y = x - 1$, which we expect to simplify the equation due to the coefficients of x^3, x^2. Then, we find $y^3 - 2021y - \sqrt{2022} = 0$, of which $\sqrt{2022}$ is a solution. $(y - \sqrt{2022})(y^2 + \sqrt{2022}y + 1) = 0$, so $r_1 - 1 = \sqrt{2022}$, $r_2 - 1, r_3 - 1 = -1(\sqrt{2022} + \sqrt{2018})/2, -(\sqrt{2022} - \sqrt{2018})/2$. Checking for the maximum, $r_1 + r_2 r_3 = 1 + \sqrt{2022} + 1 - \sqrt{2022} + 1 = \boxed{3}$.

28. Andrew is also taking a 10-question multiple-choice test. For each question, he has answer choices A, B, C and D, each with equal probability of $\frac{1}{4}$. He hates it when 3 or more questions in a row are the same answer choice. For example, he hates the key of answer choices $AAABBBCCCC$ 3 times, the key $AAAAAAAAAA$ once, and the key $AABBCCAADD$ 0 times. If the expected number of times Andrew will hate a test's key is $\frac{m}{n}$, find $100m + n$.

Answer: 2564

Proposed by Andrew Xing

For each group of 3 answer choices, from $(1-3)$ to $(8-10)$, there is a $4 \cdot (\frac{1}{4})^3$ chance that they are all equal. Since there are 8 spots, we have $4 \cdot (\frac{1}{4})^3 \cdot 8$ expected rows of 3 throughout our key.
However, this overcounts many cases where there are 4 or more in a row. To account for this, we can subtract off $4 \cdot (\frac{1}{4})^4 \cdot 7$ for the number of expected rows of 4 throughout our key.

This works because our original count for 3 rows overcounts 1 more case for each row compared to our 4 rows. For example, $AAAAA$ is counted 3 times in our 3 row case and twice in our 4 row case. So, $\frac{a}{b}$ is $32/64 - 7/64 = 25/64$ and our answer is $\boxed{2564}$.

29. Say two numbers are buddies if there exists a Pythagorean triple with the pair as distinct side lengths. A buddy sequence satisfies that each consecutive pair of integers are buddies. Given a buddy sequence of length 5 that begins with 4, find the difference between the maximal and minimal values of its last element.

Answer: 3610

Proposed by Jessica Wan

The minimal value is 3, as one can construct $4, 3, 5, 4, 3$. Now, for each consecutive pair, x, y, we have $y \leq (x^2 + 1)/2$. Clearly, 4 is only buddies with $3, 5$. Thus the second element is ≤ 5, the third $\leq (5^2 + 1)/2 = 13$, the fourth $\leq (13^2 + 1)/2 = 85$, and the last $\leq (85^2 + 1)/2 = 3613$. This is achievable with the sequence $4, 5, 13, 85, 3613$. The desired difference is $\boxed{3610}$.

30. In acute triangle ABC, let D, E be the projections of B, C onto $\overline{AC}, \overline{AB}$, respectively, and let F denote the midpoint of $\overline{BC}$. Suppose that B lies on the $F-$midline in $\triangle DEF$. Given that $DF = 4$ and the area of $\triangle ABC$ is 32, compute AF^2.

Answer: 68

Proposed by Tom Zhang

Claim: $\triangle CAF$ is isosceles.(2 proofs will be presented)

Proof 1: Because of the three tangent lemma, we get $BD \perp AC; AE \perp BC$. Then, let (F) be the point circle centered at F. We see that $FD = FE; GD = GF = FH = HE, GD^2 = GF^2 = HF^2 = HE^2$, which means that AH is the radical axis of $(CDE), (F)$. Then, we drop $CP \perp AB$, P is in AB, quadrilateral $CDPB$ is cyclic .

Since AH is the radical axis of $(CDE), (F)$, we have that $AD \cdot AC = AF^2 = \frac{AB^2}{4}$, then, since AC is the radical axis of $(CDE), (CDPB), AD \cdot AC = AP \cdot AB$, so we have $AP \cdot AB = \frac{AB^2}{4}, AP = \frac{AB}{4}$, telling that $AP = PF$, the claim is done.

Proof 2:Let ED meets the extension of BA at X. As G, H are the midpoints of FD, FE, A is the midpoint of XF. Drop $CP \perp AB$ we can see $(X, P; A, B)$ is a harmonic bundle which tells that $\frac{XA}{XB} = \frac{AP}{PB} = \frac{AF}{3AF} = \frac{1}{3}$, $AP = PF$.

Since $\triangle ADB$ is an right triangle, $DF = AF = BF = 4, AB = 8$, so the height of the triangle, $CP = \frac{32 \cdot 2}{8} = 8$. Then, $AF = AC = \sqrt{2^2 + 8^2} = 2\sqrt{17}$, so the answer is $\boxed{68}$.

31. When doing math, Yuhan realized that the $3-4-5$ and $13-14-15$ triangles are special! He calls a triangle *super* if its sides are consecutive positive integers and its area is an integer. Find the sum of all possible areas of super triangles that have areas less than 20000.

Answer: 17556

Proposed by Tom Zhang

According to Heron's formula, the area of the triangle is $\frac{\sqrt{3n \cdot (n+2) \cdot (n-2) \cdot n}}{4} = k, k \in N^*$

Expand it and we have $3n^2(n^2-4) = 16k^2$, we can see that n must be an even number, let $n = 2p, 3 \cdot 4p^2(4p^2-4) = 16k^2, 3p^2(p^2-1) = k^2$

Then, discover that k must be a multiple of p, let $k = xp$, then we leave $3p^2 - 3 = x^2$, let $x = 3s$ this time and we have $3p^2 - 3 = 9s^2, p^2 - 3s^2 = 1$, which is a Pell's equation, $(2, 1)$ is a fundamental solution for the equation. The solution is then $(n, k) = (2p, 3sp)$, which $p_i + \sqrt{3}s_i = (2+\sqrt{3})^i$

$i = 1, n = 4$, the area is 6

$i = 2, n = 14$, the area is 84

$i = 3, n = 52$, the area is 1170

$i = 4, n = 194$, the area is 16296

Thus, the sum of all possible areas is $6 + 84 + 1170 + 16296 = \boxed{17556}$

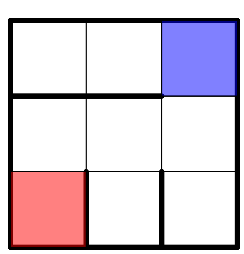
32. Consider a circle Ω with point P outside the circle. Let the tangents from P intersect the circle at A, B. Let M be the midpoint of $\overline{AP}$ and let BM intersect Ω a second time at N. If $PN = 7$, and the length of MN can be expressed as $\frac{m}{n}$, what is $100m + n$?

Answer: 702

Proposed by Xuzhou Ren

Let line PN intersect segment AB at X and Ω again at D. Since AB is the polar of P with respect to Ω, (P, N, X, D) is a harmonic bundle. By the pencil through B, this means that $(P, M, A, PA \cap BD)$ is harmonic. However, $PM = MA$, so $PA || BD$. Therefore, triangles PMN and DBN are similar and $\frac{PN}{MN} = \frac{DN}{BN} = \frac{PD}{MB}$. By power of a point, $MN \cdot MB = AM^2$ and $PN \cdot PD = 4AM^2$. Dividing out we get that $\frac{MN \cdot MB}{PN \cdot PD} = \frac{MN^2}{7^2} = \frac{1}{4}$, so $MN = \frac{7}{2} \longrightarrow \boxed{702}$

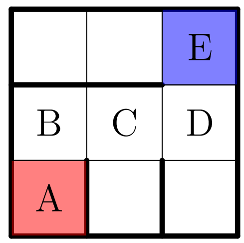
33. Charmander starts in the red cell of the grid shown below. Every minute, Charmander moves to a random adjacent cell not blocked off by a bold line. What is the expected number of moves it takes for Charmander to get to the blue cell?



Answer: 22

Proposed by Aaron Hu

Let A denote the red cell, B denote the cell above of A, C denote the cell to the right of B, D denote the cell to the right of C, and E denote the cell above D. Note that Charmander must pass through all of A, B, C, D, E.



Note that Charmander's first move must be to move up one cell, so the expected number of moves to get from A to B is 1.

Let e_1 denote the expected number of moves it takes to get from B to C. Then we have

$$e_1 = \frac{1}{2}(1) + \frac{1}{2}(e_1 + 2) \implies e_1 = 3.$$

Let e_2 denote the expected number of moves it takes to get from C to D. Then we have

$$e_2 = \frac{1}{3}(1) + \frac{1}{3}(e_2 + 2) + \frac{1}{3}(e_2 + 4) \implies e_2 = 7.$$

Finally, let e_3 denote the expected number of moves it takes to get from D to E. Then we have

$$e_3 = \frac{1}{3}(1) + \frac{1}{3}(e_3 + 2) + \frac{1}{3}(e_3 + 8) \implies e_3 = 11.$$

Hence, the answer is $1 + 3 + 7 + 11 = \boxed{22}$.

34. Consider triangle ABC with circumcircle ω and $AC > BC$. The tangent to ω at B intersects AC at D. Suppose that there exists a point M on ω such that $AM \parallel BD$ and $CM \parallel AB$. Given that DM is tangent to ω and $BC = 6$, the area of $\triangle MBD$ can be expressed as $m\sqrt{n}$ for positive integers m, n, with n square-free. Compute $100m + n$.

Answer: 1807

Proposed by Tom Zhang

Angle chasing gives $\angle MAB = \angle DBM = \angle BMA$, so $BM = BA$. Note that $ABCM$ is harmonic, so $AB \cdot CM = BC \cdot AM = 36$. Then from Ptolemy,

$$BM^2 = AC \cdot BM = AB \cdot CM + BC \cdot AM = 36 + 36 = 72,$$

so $BM = 6\sqrt{2}$. Note that both $\triangle MBD$ and $\triangle MAB$ are isosceles and $\angle MBD = \angle MAB$, implying that they are similar. Then

$$DM = DB = BM^2/AM = 72/6 = 12,$$

from which we can compute the area of $\triangle MBD$ to be $18\sqrt{7}$, for an answer of $\boxed{1807}$.

35. Suppose the Fibonacci sequence begins with $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Find the minimal n such that
$$\sum_{i=2}^{n} \frac{1}{F_{i-1}F_{i+1}} \geq \frac{4095}{4096}.$$

Answer: 10

Proposed by Jessica Wan

Note that $1/F_{i-1}F_{i+1} = F_i/F_{i-1}F_iF_{i+1} = (F_{i+1} - F_{i-1})/F_{i-1}F_iF_{i+1} = 1/F_{i-1}F_i - 1/F_iF_{i+1}$. Thus, $\Sigma_{i=2}^{n} 1/F_{i-1}F_{i+1} = 1/F_1F_2 - 1/F_nF_{n+1} = 1/2 - 1/F_nF_{n+1}$. We desire n such that $1/F_nF_{n+1} \leq 1/4096$. Cursory checking of small indices reveals $F_9F_{10} < 2^{12} \leq F_{10}F_{11}$. Thus, $\boxed{10}$ is our answer.

36. Karthik is competing in an hour-long pie-eating contest. He finishes eating a pie only when the amount of time passed since the contest started (in seconds) is a positive perfect square. At some random time within the hour, Karthik is scared away by Alex and stops eating pies. Given that the expected value of the number of pies Karthik eats before he is scared off can be expressed as $\frac{m}{2} - \frac{1}{n}$ for positive integers m, n, compute $100m + n$.

Answer: 8260

Proposed by Aaron Hu

Let k denote the number of pies Karthik eats before being scared off. Then the random time Alex shows up at must lie on the interval $[k^2, (k+1)^2)$, where the numbers represent the amount of time elapsed since the contest started in seconds. Then

$$\begin{aligned}\mathbb{E}(k) &= \sum_{k=0}^{59} k\left(\frac{(k+1)^2 - k^2}{60^2}\right) = \frac{1}{60^2}\sum_{k=0}^{59}[2k^2 + k] = \frac{1}{60^2}\left(\frac{59 \cdot 60 \cdot 119}{3} + \frac{59 \cdot 60}{2}\right) \\ &= \frac{59(119 \cdot 2 + 3)}{6 \cdot 60} = \frac{59 \cdot 241}{360}.\end{aligned}$$

Substituting $x = 60$ gives

$$\frac{59 \cdot 241}{360} = \frac{(x-1)(4x+1)}{6x} = \frac{2}{3}x - \frac{1}{2} - \frac{1}{6x} = \frac{79}{2} - \frac{1}{360},$$

so the answer is $100 \cdot 79 + 360 = \boxed{8260}$.

37. In chess, pawns are worth 1 point, rooks are worth 5 points, and queens are worth 9 points. How many ways can Andrew gather an assortment of pawns, rooks, and queens such that the total value of his pieces is 450 points? Assume that pieces of the same type are indistinguishable and that he does not necessarily have a piece of every type.

Answer: 2326

Proposed by Aaron Hu

Let p, q, r denote the number of pawns, queens, and rooks Andrew has, respectively. Then we want to find the number of ordered triples of nonnegative integers (p, q, r) such that

$$p + 9q + 5r = 450.$$

This is equivalent to finding the number of ordered pairs of nonnegative integers (x, y) such that $9x + 5y \leq 450$, as there exists the bijection

$$(p, q, r) \leftrightarrow (450 - 9x - 5y, x, y).$$

Note that the number of such pairs is just the number of lattice points contained in or on the boundary of the triangle with vertices $(0,0)$, $(50,0)$, $(0,90)$ in the coordinate plane. We find that the number of boundary points is $50+90+10=150$, so from Pick's theorem, the number of interior points is

$$\frac{1}{2}(50)(90)-\frac{1}{2}(150)+1=2250-75+1=2176,$$

so the answer is $2176+150=\boxed{2326}$.

38. Tom is bored and decides to repeatedly write the digits of 2023 periodically until his number is divisible by 2023^2 (the first 5 versions of his number are $2, 20, 202, 2023, 20232$). How many digits is his final number?

Answer: 816

Proposed by Aaron Hu

In order for 2023 to divide the final number, the number of digits must be divisible by 4, so suppose the final number has $4n$ digits. Then the number is equal to

$$\underbrace{20232023\ldots2023}_{n\ 2023\text{'s}}=2023(10^{4n-4}+10^{4n-8}+\cdots+10^4+1)=2023\cdot\frac{10^{4n}-1}{10^4-1}.$$

Note that $\gcd(2023,10^4-1)=1$, so we need only find the smallest positive integer n such that $2023\mid 10^{4n}-1$, or equivalently, $7\mid 10^{4n}-1$ and $17^2\mid 10^{4n}-1$.

Note that $\text{ord}_7(10)\mid 7-1=6$ and $7\nmid 10^2-1, 10^3-1$, so this order equals 6, implying that $6\mid 4n$. Additionally, $\text{ord}_{17}(10)\mid 17-1=16$ and $7\nmid 10^8-1$, so this order equals 16. We can check that $\nu_{17}(10^{16}-1)=1$, so from LTE, we have $16, 17\mid 4n$. This implies that $4n=3\cdot16\cdot17=\boxed{816}$, and we are done.

39. Consider $\triangle ABC$ with $\angle A=90^\circ$ and $AB=10$. Let D be a point on $\overline{AB}$ such that $BD=6$. Suppose that the angle bisector of $\angle C$ is tangent to the circle with diameter $\overline{BD}$ and say it intersects $\overline{AB}$ at point E. Given that BE can be expressed as $\frac{a+b\sqrt{c}}{d}$ for positive integers a,b,c,d with $\gcd(a,b,d)=1$ and c square-free, compute $1000a+100b+10c+d$.

Answer: 40000

Proposed by Tom Zhang

Let the center of the circle be O, let $OE=x, BE=3+x, AE=7-x$, call the tangency point be N. As the problem states, $\angle ECB=\angle ACE$, as $\triangle CAE\sim\triangle ONE.\angle NOE=\angle ACE$, we call that $\angle NOE=\angle ACE=\theta$, $\cos\theta=\frac{3}{x}$, angle bisector theorem implies that $\frac{AC}{BC}=\frac{7-x}{3+x}=\cos 2\theta=\frac{18-x^2}{x^2}$, solve the equation and we attain $x=\frac{3\sqrt{69}+9}{10}$, $BE=3+x=\frac{3\sqrt{69}+39}{10}$ leads to $\boxed{40000}$

40. How many ways are there for a rook to return to its original square on a 8×8 chessboard in 6 moves if it starts on a corner? A "move" counts as shifting the rook by a positive number of squares on the board along a row or column. Note that the rook may return back to its original square during an intermediate step within its 6-move path.

Answer: 127904

Proposed by Aaron Hu

Let the rook be in the bottom left corner. Now, consider the following diagram.

6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
14	6	6	6	6	6	6	6

Note that the first $n-2$ moves are irrelevant. The $n-1^{\text{st}}$ move must get the rook into position to head back on the n^{th} move. The above diagram shows the number of ways for the rook to make the $n-1^{\text{st}}$ move if it were on that cell.

Let a_n denote the number of ways to get to the cell marked 14 in n moves. Let b_n denote the number of ways to get to a cell marked 6 in n moves. Let c_n denote the number of ways to get to a cell marked 2 in n moves. Clearly, $c_n = 14^n - a_n - b_n$. Then we have the recursions

$$a_n = b_{n-1}, \quad b_n = 14a_{n-1} + 6b_{n-1} + 2c_{n-1}.$$

Clearly, $a_1 = 0$, $a_2 = b_1 = 14$, so we have

n	1	2	3	4
a_n	0	14	84	896
b_n	14	84	896	10080
c_n	0	98	1764	27440

Then we have $a_6 = b_5 = 14 \cdot 896 + 6 \cdot 10080 + 2 \cdot 27440 = \boxed{127904}$.

41. Consider the following equation:

$$\left(\frac{3}{8}x^3 + 1\right)^3 = \left(\frac{8}{3}(x^2 - 1)\right)^2.$$

This has two real solutions that are greater than 1. One of them is $x = 2$. The other one is $x = \frac{a+\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$. Compute $100a + 10b + c$.

Answer: 233

Proposed by Karthik Vedula

Take the positive 6th root of both sides. This gives

$$\sqrt{\frac{3}{8}x^3 + 1} = \sqrt[3]{\frac{8}{3}(x^2 - 1)}.$$

Note that these functions are inverses (when defined in $x > 1$):

$$y = \sqrt{\frac{3}{8}x^3 + 1} \implies y^2 - 1 = \frac{3}{8}x^3 \implies x = \sqrt[3]{\frac{8}{3}(y^2 - 1)}.$$

Also, note that both functions are increasing on $x > 1$. This means that when they intersect, they must intersect on the line $y = x$ (since the graphs are reflections over this line). Now, we can solve a standard cubic:

$$\sqrt{\frac{3}{8}x^3 + 1} = x \implies 3x^3 - 8x^2 + 8 = 0 \implies (x-2)(3x^2 - 2x - 4) = 0 \implies x = 2, \frac{1 \pm \sqrt{13}}{3}.$$

The solution that we are looking for is $\frac{1+\sqrt{13}}{3}$, and our answer is $100 + 130 + 3 = \boxed{233}$.

42. Consider polynomial $P(n) = n^3 - 8n^2 + 4n + 12$ with real roots a, b, c. Let Q denote a monic cubic polynomial with roots x, y, z with

$$P(x) + Q(a) = P(y) + Q(b) = P(z) + Q(c) = 0,$$

and P, Q having no roots in common. Given that $x + y + z = 6$ and $x^3 + y^3 + z^3 = 69$, compute $Q(0)$.

Answer: 81

Proposed by Aaron Hu

Note that

$$P(x) = (x-a)(x-b)(x-c) = -(a-x)(a-y)(a-z) = -Q(a),$$

and since $a \neq x$, we have $(x-b)(x-c) = (a-y)(a-z)$. Similarly,

$$(y-a)(y-c) = (b-x)(b-z), \quad (z-a)(z-b) = (c-x)(c-y),$$

so summing gives

$$\begin{aligned} \sum_{\text{cyc}} \left[x^2 - (b+c)x + bc\right] &= \sum_{\text{cyc}} \left[a^2 - (y+z)a + yz\right] \\ \implies \sum_{\text{cyc}} \left[x^2 + bc\right] &= \sum_{\text{cyc}} \left[a^2 + yz\right] \\ \implies \sum_{\text{cyc}} \left[x^2 - yz\right] &= \sum_{\text{cyc}} \left[a^2 - bc\right]. \end{aligned}$$

Then from Vieta,

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \\ &= 6(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= 6(8^2 - 3 \cdot 4) \\ &= 312, \end{aligned}$$

so

$$Q(0) = -xyz = \frac{312 - 69}{3} = \boxed{81},$$

as desired.

43. The integers from 1 to 2022 are placed around a circle in this order. Bob the beaver sits at 1 on the circle. Each minute, he moves left or right to the next number with equal probability. What is the expected number of minutes needed for Bob to visit each number at least once?

Answer: 2043231

Proposed by Jessica Wan

Note that at any point the set of numbers Bob has visited forms a continuous arc of the circle. Thus, we simply find the expected number of moves to reach $i + 1$ distinct numbers starting from the boundary of an arc of i numbers. Using states, one can find this value is equal to i. Thus, the expected number to visit all 2022 distinct numbers is $1 + 2 + \cdots + 2021 = \boxed{2043231}$.

44. Andrew and Tiger and playing a game. They start with the number $n^2 + 1$ for some positive integer n and take turns subtracting nonzero perfect squares from the number such that the number is always nonnegative. The player that reduces the number to 0 wins. What is the minimum starting number such that Andrew wins given that he goes first and both players play optimally?

Answer: 26

Proposed by Aaron Hu

Consider the function f where for positive integer k, $f(k)$ equals 1 if Andrew wins with starting number k and 0 otherwise. We can easily verify that

$$f(1) = 1, \quad f(2) = 0, \quad f(3) = 1, \quad f(4) = 1, \quad f(5) = 0.$$

I claim this pattern repeats every five integers all the way up to $f(20)$. Induct on groups of five integers. The base case has been established, so suppose that the pattern repeats until $5k$ for some integer $1 \leq k < 4$. Since we are dealing with numbers less than 5, the perfect squares we can subtract are equivalent to either 1 or 4 modulo 5.

- For $5k + 1$, Andrew can subtract 1 to force a losing number on Tiger, so $f(5k + 1) = 1$.
- For $5k + 2$, the number is either 1 or 3 modulo 5 after Andrew moves, which is winning for Tiger, so $f(5k + 2) = 0$.
- For $5k + 3$, Andrew can again subtract 1 to win, so $f(5k + 3) = 1$.
- For $5k + 4$, Andrew can subtract 4 to win, so $f(5k + 4) = 0$.
- For $5k + 5$, after Andrew's turn, the number is either 1 or 4 modulo 5, which is winning for Tiger in both cases, so $f(5k + 5) = 0$.

This completes the induction. Therefore 2, 5, 10, and 17 are all losing for Andrew. Now consider starting with 26. Andrew can subtract 4^2 to give 10 to Tiger, which is losing for him, so the answer is $\boxed{26}$.

45. Consider $\triangle ABC$ with M denoting the midpoint of $\overline{BC}$. Let the B and C angle bisectors intersect the circle centered at M passing through B, C a second time at U, V, respectively. Let $P = BV \cap CU$, and let the circumcircle of $\triangle UVM$ intersect $\overline{BC}$ a second time at D. Given that $BD = 10$, $CD = 21$, and $\angle A = 60^\circ$, find PD^2.

Answer: 675

Proposed by Aaron Hu

From Iran Lemma, $\overline{BU} \perp \overline{PC}, \overline{CV} \perp \overline{PB}$. Then $\odot(UVM)$ is just the nine-point circle of $\triangle PBC$, so $\overline{PD}$ is the $P-$altitude in $\triangle PBC$, and in particular, the incenter I of $\triangle ABC$ is the orthocenter of $\triangle PBC$.

Note that D is the $A-$intouch point of $\triangle ABC$, so $s - b = 10, s - c = 21$, implying that $a = 31$. From Law of Cosines, $b^2 + c^2 - bc = 961$, and since $b - c = 21 - 10 = 11$, we have

$$bc = b^2 + c^2 - bc - (b - c)^2 = 961 - 121 = 840.$$

Then the area of $\triangle ABC$ is $\frac{1}{2} bc \sin 60^\circ = 210\sqrt{3}$. We also have

$$(b + c)^2 = (b - c)^2 + 4bc = 121 + 4 \cdot 840 = 3481,$$

so $b + c = 59$. Then $s = (59 + 31)/2 = 45$, so the inradius of $\triangle ABC$ is $210\sqrt{3}/45 = 14\sqrt{3}$. Let I' denote the reflection of I over $\overline{BC}$. Then $I' \in \odot(PBC)$, so from Power of a Point,

$$210 = BD \cdot DC = PD \cdot DI' = \frac{14}{\sqrt{3}} \cdot PD,$$

so $PD = 15\sqrt{3}$, and the answer is $\boxed{675}$.

46. A function $f : \{0, 1, 2, ..., 2021\} \to \{0, 1, 2, ..., 2021\}$ is *strong* if it satisfies

$$f(x+y) \equiv f(x)\,f(y) \pmod{2022}$$

whenever f is defined on x, y, $x+y$. Find the number of strong functions.

Answer: 2023

Proposed by Jessica Wan

Note that for natural n, $f(n) \equiv f(1)^n \mod 2022$. Also, $f(x) \equiv f(x)f(0) \mod 2022$, and $f(0) \equiv 0, 1, -1$. However, $f(0)^2 \equiv f(0) \mod 2022$, so $f(0) \neq -1$.

Thus, if $f(0) \equiv 0$, $f(x) \equiv 0$ for all x. Else, if $f(0) \equiv -1$, then $f(x) \equiv 0$ for all positive x.

Finally, when $f(0) \equiv 1$, $f(1)$ may take any value for 2022 solutions, for $\boxed{2023}$ functions in all.

47. The *score* of a nonnegative integer n is defined as $\sum_{i=0}^{9} i f_i(n)^2$, where $f_i(n)$ denotes the number of times the digit i appears in the decimal representation of n. For example, the score of 12341 is $1 \cdot 2^2 + 2 \cdot 1^2 + 3 \cdot 1^2 + 4 \cdot 1^2 = 15$. Compute the floor of the average score of all integers from 0 to $10^{2022} - 1$.

Answer: 1848006

Proposed by Jessica Wan

Note that the digit 0 does not contribute, and for each digit $1 \le i \le 9$, there are exactly $\binom{2022}{j} \cdot 9^{2022-j}$ integers from 0 to $10^{2022} - 1$ in which it appears j times.

Thus, the total sum of scores can be calculated as the total contribution of the digits $1-9$, and is equal to:

$$\sum_{i=1}^{9} i \left(\sum_{j=0}^{2022} j^2 \binom{2022}{j} 9^{2022-j} \right) = 45 \cdot \sum_{j=1}^{2022} 2022j \binom{2021}{j-1} 9^{2022-j}$$

$$= 45 \cdot 2022 \cdot \left(\sum_{j=1}^{2022} \binom{2021}{j-1} 9^{2022-j} + \sum_{j=1}^{2022} \binom{2021}{j-1} \cdot (j-1) 9^{2022-j} \right)$$

$$= 45 \cdot 2022 \cdot \left((9+1)^{2021} + \sum_{i=2}^{2022} 2021 \binom{2020}{j-2} 9^{2022-j} \right)$$

$$= 45 \cdot 2022 \cdot \left(10^{2021} + 2021 \cdot 10^{2020} \right).$$

Thus the average score is equal to:

$$\frac{45 \cdot 2022 \cdot 2031 \cdot 10^{2020}}{10^{2022}} = \frac{45 \cdot 2022 \cdot 2031}{100} = 1848006.9,$$

for a final answer of $\boxed{1848006}$.

48. In triangle ABC, points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $DE \parallel BC$. Let CD and BE intersect $\odot(ABC)$ a second time at F and G, respectively. Let the circumcircles of $\triangle ADF$ and $\triangle AEG$ intersect a second time at Q, and let AQ intersect BC at P. Suppose that $AD = 6$, $BD = 2$, and $AE = 9$. Given that the length of CD is an integer and is minimized, AP^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

Answer: 28703

Proposed by Tom Zhang

Claim: P is the midpoint of BC.

Proof: Extend DE to meet (ABC) at K. As $DE||BC, \angle GEK = \angle GBC = \angle GFD$, so $GFDE$ is a cyclic quadrilateral. As GE, AQ, FD are radical axes of $(FDEG), (AEG); (ADF), (AEG); (FDEG), (ADF)$, so they meet at the same point.

Now, according to Ceva theorem, $\frac{BP}{PC} \cdot \frac{CE}{EA} \cdot \frac{AD}{DB} = 1$, as DE is parallel to BC, $\frac{CE}{EA} \cdot \frac{AD}{DB} = 1, BP = PC$, the claim is done.

Assume $BP = PC = x$, according to Stewart's theorem, we can attain $6 \cdot 4x^2 + 12^2 \cdot 2 = 8(12 + CD^2)$ and $4x^2 \cdot 9 + 8^2 \cdot 3 = 12(27 + BE^2)$.

Combine those two equations together, we can get $3x^2 + 24 = CD^2, 3x^2 - 11 = BE^2, CD^2 - BE^2 = 35$ As CD has to be an integer, when $CD = 6, BE = 1, AB + BE = AE$, so it can't work, so CD has to be equal to 7

This time, $x^2 = \frac{25}{3}$, apply Stewart again, $64x + 144x = 2x(x^2 + AP^2), AP^2 = 104 - x^2 = \frac{287}{3}$, leads to $\boxed{28703}$

49. In a small country with 20 cities, a set of one-way roads (currently not directed) is constructed such that each road starts at one city and ends at a different city with no more than 1 road between 2 cities. Each city belongs to 1 of k political parties, and cities that are connected by a road cannot belong to the same political party. Each city has exactly 18 roads to other cities. Let an *orientation* of these roads be adding a direction to each road. Let an *oriented path* be a path from a city A to another city B that consists of roads all oriented from the city closer to A to the city closer to B, with no city in the path being visited twice. Let the *length* of this path be the number of roads in it. Let the oriented path with maximal length in an orientation be P. What is the minimum length of P over all possible orientations of the roads in this country?

Answer: 6

Proposed by Xuzhou Ren

et the cities be vertices and the roads be unoriented edges of a graph G. Let the set of all orientations of G be $O(G)$. Let the chromatic number (least number of colors needed to color the graph such that any adjacent vertices have different colors) of G be $\chi(G)$.

The main claim is the Gallai-Roy-Vitaver theorem:

$$\chi(G) = 1 + min_{D \in O(G)}(P)$$

Proof: We wish to prove that:

1) $\exists D \in O(G)$ such that $\chi(G) \geq 1 + P$

2) $\forall D \in O(G), P \geq \chi(G) - 1$

Proof of 1: Let $\chi(G) = K$, pick a proper K-coloring of G. Orient each edge from the vertex with smaller color to the one with larger color.

Claim: $P \leq K - 1$

Proof: Let the vertices that belong to P be $a_1, a_2, \cdots$ and the orientation go away from a_1, we get that $color(a_1) < color(a_2) < \cdots$ since colors increase along the oriented path. Therefore $P \leq K - 1$.

Proof of 2: Fix some $D \in O(G)$. Construct a graph with vertex set $V = V(G)$ and start with no edges. Then, add edges from D as long as the added edge does not create an oriented cycle. Once we cannot add any more edges, call this graph D'. Color each vertex in D' $color(v) = 1+$max length of an oriented path in D' ending at v. We claim that this coloring is proper.

Proof: Since adding any edge in G but not in D' will create an oriented cycle, we can always find an oriented path between any two vertices in D'.

It suffices to show that the colors are increasing along any oriented path in D', or that if an edge is oriented from $u \to v$, then $color(u) < color(v)$.

This is also equivalent to the max length of an oriented path in D' ending at u is less than the max length of an oriented path in D' ending at v.

Let u and v lie on an oriented path T of maximal length that starts at some w. Then, since the orientation is from $u \to v$, letting $d(a,b)$ be the distance between a, b $d(u,w) < d(v,w)$. Since there are no oriented cycles, v cannot appear any earlier on that path because doing so would create an oriented cycle starting at v, traveling along the path to u, and then going back to v. We are done.

The new problem All that remains is to calculate the minimum possible $\chi(G)$ over all possible graphs. We claim that this is $\chi(G) = 7$.

Observe that our graph is. $K21$ minus a collection of disjoint cycles. We claim that the optimal collection of cycles in order to minimize the chromatic number is a disjoint union of 7 triangles(3-cycles).

Proof: Let the $optimality$ of a cycle be

$$\frac{length\ of\ the\ cycle}{minimum\ number\ of\ colors\ needed\ tocolor\ the\ cycle}.$$

We are trying to find the cycle with the largest $optimality$. Observe that 1 color is enough to color a triangle. In general, it's not hard to we can color the cycle in $\lfloor \frac{n}{2} \rfloor$ by just taking adjacent vertices. This is also the minimum because any $n - cycle$ contains a $K \lfloor \frac{n}{2} \rfloor$. Therefore, it's not hard to see that the most optimal cycle is the triangle.

Since the optimality counts how good the cycle is at saving colors, choosing all triangles will be the best. Therefore, we can let the vertices of each triangle be the same color, and since there are 7 triangles $\chi(G) = 7$. This is the minimum because by taking 1 vertex from each disjoint triangle we end up with a copy of K_7.

So $min(P) = \chi(G) - 1 = 6$.

50. In $\triangle ABC$, we have $AB = 5$, $BC = 6$, and $CA = 7$. A variable point P lies on arc BC not containing A of the circumcircle of $\triangle ABC$. Let D and E be the incenters of ABP and ACP, respectively. As P varies on arc BC, the midpoint of $\overline{DE}$ always lies on a fixed circle with area $\frac{m}{n}\pi$, where m, n are coprime, positive integers. Compute $100m + n$.

Answer: 31532

Proposed by Karthik Vedula

Let I denote the incenter of ABC and F denote the incenter of BCP. The main claim is that $IDFE$ is a rectangle, which is the Japanese theorem for cyclic quadrilaterals. This means the midpoint of $\overline{DE}$ is the midpoint of $\overline{IF}$.

I is fixed as P varies, and by Fact 5, F lies on a circle centered at the midpoint of arc BC that contains A (which we will call M) passing through B and C. This means the midpoint of $\overline{IF}$ lies on the circle which is the image of the aforementioned circle dilated centered at I with factor $\frac{1}{2}$. This means the radius of our desired circle is $0.5MB = 0.5MC$.

From here, we use trigonometry. Note that if $x = BM$, then by LoC, we have

$$\cos BMC = \cos BAC \implies \frac{x^2 + x^2 - 6^2}{2x^2} = \frac{5^2 + 7^2 - 6^2}{2 \cdot 5 \cdot 7} \implies x = BM = \sqrt{\frac{315}{8}}$$

The area of our desired circle is $0.25BM^2\pi = \frac{315}{32}\pi$, so our answer is $100 \cdot 315 + 32 = \boxed{31532}$.