Florida Student Association of Mathematics

2022 Winter Florida Online Math Open

December 26, 2022 - January 1, 2023

Problem Contributors

Test Coordinators

- Karthik Vedula (`@kvedula#4055`)
- Alex Li (`@idk#5799`)
- Aaron Hu (`@Dyno#4444`)

Curriculum Committee

- Rui Jiang
- Sailalitha Kodukula
- Ahan Mishra
- Yuhan Niu
- Xuzhou Ren
- Anagh Sangavarapu
- Jessica Wan
- Andrew Xing
- Tom Zhang

Thank you to everyone who submitted problems!

Rules

1. **1.** This is a 50 problem, team-based competition that takes place over four days. Answers must be submitted before 7PM EDT on January 1st.
2. **2.** This is a team event; collaboration with your teammates is encouraged. Discussion about the contest or its problems is not allowed between members of different teams or anyone not competing.
3. **3.** Each correct answer is worth 1 point. Incorrect or omitted answers are both worth 0 points.
4. **4.** The difficulty of each question will be determined by how many teams correctly answer it. Questions with fewer solves will be considered more difficult. Ties will be broken by considering the most difficult question each team correctly answers. The team that correctly answered the more difficult question wins the tiebreaker. If necessary, the process will move to the second most difficult question each team correctly answers and so on.
5. **5.** You may use four function calculators. All problems on this test can be solved without a calculator. Drawing aids such as graph paper, rulers, compasses, and protractors are permitted, but electronic programs such as graphing calculators, Desmos, or Geogebra are not.
6. **6.** **All answers are non-negative integers**. Any non-integer quantities will be converted to integers by instructions in the problem. Fractions of the form $\frac{m}{n}$ and radicals of the form $p\sqrt{q}$ are assumed to be in simplest form (that is, $\gcd(m,n) = 1$, q is squarefree, and all variables are positive integers). Isolated radicals of the form $\sqrt{k}$ do not necessarily follow this rule.
7. **7.** Make sure to check both the `#flomo-updates` channel in Discord and emails from `flsamflorida@gmail.com` for any clarifications or updates. If you have any questions, please contact the testing coordinators at `flsamflorida@gmail.com` or `alexander.limy@gmail.com` or through Discord DMs to the given tags.
8. **8.** Potato. That is all.

In addition, the following approximations and definitions may be useful:

The floor function, $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

The ceiling function, $\lceil x \rceil$, is the least integer greater than or equal to x.

$$\pi \approx 3.14159265359$$

$$e \approx 2.71828182846$$

$$693e + 44\pi \approx 2022$$

Problems

1. A toddler's toy set includes three blocks in the shape of a rectangular prism, a triangular prism, and a cylinder. A palette contains holes with the shapes of a rectangle, triangle, and circle. The rectangular prism can fit in any of the three holes, the triangular prism can only fit in the trianglular hole, and the cylinder can only fit in the circular hole. How many ways are there to assign distinct blocks to each hole so that at least two blocks fit their hole?

2. Yuhan chooses a random positive integer divisor d of 8658. Let p be the probability that d is a palindrome with more than one digit. Compute $420p$.

3. Compute the number of terms that have rational coefficients in the expansion of $(x\sqrt[8]{4}+y\sqrt[4]{8})^{1000}$ when like terms are combined.

4. Consider triangle ABC. Let the circle with diameter BC intersect AB at D and AC at E. Let BE and CD intersect at X. Let line AX intersect BC at F. If $BF = 2$, $CF = 7$, and $AF = 7$, what is the length of XF?

5. A knight at point (x, y) in the coordinate plane can move to either $(x+1, y+2)$ or $(x+2, y+1)$. How many ways can a knight move from $(1, 0)$ to $(20, 23)$?

6. Triangle ABC has sides of length 49, 50, and 51. Points D and E are chosen on distinct sides of ABC such that $\overline{DE}$ bisects the area of the triangle. What is the minimum possible integer value of the sum of the distances from D and E to the common vertex of their edges?

7. Aaron is taking a 10-question multiple-choice test. For each question, he has answer choices A, B, C, and D. Aaron hates a key when its answer choice switches. For example, he hates the key of answer choices $AAABBBCCCC$ 2 times and the key $AAAAAAAAAA$ 0 times. If the expected number of times Aaron will hate a test's key is $\frac{m}{n}$, where m, n are relatively prime positive integers, find $100m + n$.

8. Jessica has four sticks with lengths 2021, 2022, 2023, and 2026. First, she forms a cyclic quadrilateral with area A using the four sticks. She then joins the two sticks with lengths 2021 and 2023 together to form a longer stick. She forms a triangle with area B using the three sticks she now has. Given that $\frac{A}{B}$ can be expressed as $\frac{m}{n}$ for coprime positive integers m, n, compute $100m + n$.

9. There are two distinct solutions α, β to the equation $4\cos x + 3\sin x + \frac{\sqrt{69-\sqrt{420}}}{2} = 0$ for $0 < x < 2\pi$. If the value of $\cos(\alpha+\beta)$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, what is $100m + n$?

10. Consider the function $f(x) = \lfloor \pi x \rfloor - \lceil ex \rceil$. Let m and n denote the number of positive integer solutions to $f(x) = f(x-1)$ and $f(x) = f(x-1) + 2$, respectively, where $1 \le x \le 1000$. Compute $m - n$.

11. Call a nonnegative integer *cute* if the sum of its digits is a multiple of 7. Tiger lists all cute numbers between 0 and 2023, inclusive. How many numbers does he list?

12. Consider the function $f(x) = x^2 + mx + n$, where m and n are real constants. How many ordered pairs (m, n) are there such that there exist distinct integers a, b satisfying $f(a) = f(b) = 69$ and $f(a+b) = 420$?

13. If the area of a triangle with distinct side lengths $8, 15x$, and $17x$ is maximized at $x = \frac{\sqrt{m}}{n}$ for positive integers m, n, and m square-free, find $100m + n$.

14. Consider a segment of length 1. Five points are selected randomly and independently along the length of the segment, partitioning the segment into six sub-segments. The probability that the resulting sub-segments can connect to form a hexagon can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

15. Consider $x, y, z \in R$ such that $\frac{(x+2)^2}{2} + \frac{(y+9)^2}{9} + \frac{(z+25)^2}{25} = 1$. What is the difference between the maximum and minimum possible values of $x + y + z$?

16. How many ways are there to tile a 10×18 board with L-shaped tetrominoes?

17. A random point is selected with coordinates (x, y) satisfying $0 \leq x, y \leq 1$. The probability that point $(4\sqrt{3}x + 6y, 2x + 6\sqrt{3}y)$ lies inside the circle $(x - 8\sqrt{3})^2 + (y - 8)^2 = 12$ can be expressed as $\frac{a\pi - b\sqrt{c}}{d}$ for positive integers a, b, c, d satisfying $\gcd(a, b, d) = 1$ and c square-free. Find $1000a + 100b + 10c + d$.

18. Let S be the set of all positive integers n such that $9^{3n} - 5^{2n}$ is divisible by 2^n. Find the sum of the elements in S.

19. Call a positive integer n *consistent* if the following is true: there exists a positive integer y such that if x is a positive integer where 2^{n+x} and 5^x have the same left-most digit, then this digit is equal to y. Compute the product of all consistent numbers.

20. The equation $x^4 = 4x^3 + 6x^2 + 4x + 1$ has two real solutions a, b. There exists positive rational numbers r, s such that $|a| + |b| = 2^r + 2^s$. The value of $r^2 + s^2$ is equal to $\frac{m}{n}$ for coprime, positive integers m, n. Compute $100m + n$.

21. Alex needs help evaluating this series. If the sum

$$1^2 \cdot 2! + 2^2 \cdot 3! + 3^2 \cdot 4! + \ldots + 100^2 \cdot 101!$$

can be represented as $a \cdot b! + 2$ for positive integers a, b with b maximized, find $100a + b$.

22. The only real root of the equation

$$(3x - 1)(\sqrt{9x^2 - 6x + 5} + 1) + (x - 4)(\sqrt{x^2 - 8x + 20} + 1) = 0$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

23. There are 16 pool balls in total: 1 cue ball, 1 8-ball, and 7 pairs of matching colored solid and stripe balls. A game is won if a player sinks the 8-ball. A player can only attempt to sink the 8-ball once he or she sinks all the balls of their pattern, which is chosen at the beginning of the game. A player's turn continues if they sink a ball, and ends if they fail to sink one. Assuming no misplays occur, there are n different orders of balls sunken if there are 5 turns in total and the player who chose stripes wins. Compute the remainder when n is divided by 10^4.

24. Consider the set S of all ordered quadruples (r_1, r_2, r_3, r_4) of four distinct roots of the equation

$$2x^5 - (a + 1)x^4 + (2a + 2)x^3 - (3a + 1)x^2 - a = 0,$$

where a is a constant. Find

$$\sum_{(r_1, r_2, r_3, r_4) \in S} (r_1 + r_2 - 1)(r_3 r_4 - 1),$$

where the sum is taken over all elements in S.

25. Stephen Curry has a career 3-point percentage of 42.8%. After he shoots a 3-pointer, the ball can roll an integer distance anywhere between 0 to 10 feet, with a probability proportional to $10 - d$, where d is the distance rolled. Curry comes into the gym and shoots 10 shots regardless of whether he makes them, but after 10, if he misses one he stops shooting. The expected accumulated distance the ball rolls over all of his shots can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $100m + n$.

26. Alex likes numbers that can be written as $5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}$, with a_i distinct non-negative integers. Find the sum of all the numbers Alex likes up to 2^{14}.

27. Let the roots of the polynomial $x^3 - 3x^2 - 2018x + 2020 = \sqrt{2022}$ be r_1, r_2, r_3. Compute the maximum possible value for $r_1 + r_2 r_3$.

28. Andrew is also taking a 10-question multiple-choice test. For each question, he has answer choices A, B, C and D, each with equal probability of $\frac{1}{4}$. He hates it when 3 or more questions in a row are the same answer choice. For example, he hates the key of answer choices $AAABBBCCCC$ 3 times, the key $AAAAAAAAAA$ once, and the key $AABBCCAADD$ 0 times. If the expected number of times Andrew will hate a test's key is $\frac{m}{n}$, find $100m + n$.

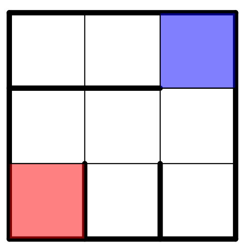
29. Say two numbers are buddies if there exists a Pythagorean triple with the pair as distinct side lengths. A buddy sequence satisfies that each consecutive pair of integers are buddies. Given a buddy sequence of length 5 that begins with 4, find the difference between the maximal and minimal values of its last element.

30. In acute triangle ABC, let D, E be the projections of B, C onto $\overline{AC}, \overline{AB}$, respectively, and let F denote the midpoint of $\overline{BC}$. Suppose that B lies on the $F-$midline in $\triangle DEF$. Given that $DF = 4$ and the area of $\triangle ABC$ is 32, compute AF^2.

31. While doing math, Yuhan realizes that the $3-4-5$ and $13-14-15$ triangles are special! He calls a triangle *super* if its sides are consecutive positive integers and its area is an integer. Find the sum of all possible areas of super triangles that have areas less than 20000.

32. Consider a circle Ω with point P outside the circle. Let the tangents from P intersect the circle at A, B. Let M be the midpoint of $\overline{AP}$ and let BM intersect Ω a second time at N. If $PN = 7$, then the length of MN can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. What is $100m + n$?

33. Charmander starts in the red cell of the grid shown below. Every minute, Charmander moves to a random adjacent cell not blocked off by a bold line. What is the expected number of moves it takes for Charmander to get to the blue cell?



34. Consider triangle ABC with circumcircle ω and $AC > BC$. The tangent to ω at B intersects AC at D. Suppose that there exists a point M on ω such that $AM \parallel BD$ and $CM \parallel AB$. Given that DM is tangent to ω and $BC = 6$, the area of $\triangle MBD$ can be expressed as $m\sqrt{n}$ for positive integers m, n, with n square-free. Compute $100m + n$.

35. Suppose the Fibonacci sequence begins with $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Find the minimal n such that

$$\sum_{i=2}^{n} \frac{1}{F_{i-1}F_{i+1}} \geq \frac{4095}{4096}.$$

36. Karthik is competing in an hour-long pie-eating contest. He finishes eating a pie only when the amount of time passed since the contest started (in seconds) is a positive perfect square. At some random time within the hour, Karthik is scared away by Alex and stops eating pies. Given that the expected value of the number of pies Karthik eats before he is scared off can be expressed as $\frac{m}{2} - \frac{1}{n}$ for positive integers m, n, compute $100m + n$.

37. In chess, pawns are worth 1 point, rooks are worth 5 points, and queens are worth 9 points. How many ways can Andrew gather an assortment of pawns, rooks, and queens such that the total value of his pieces is 450 points? Assume that pieces of the same type are indistinguishable and that he does not necessarily have a piece of every type.

38. Tom is bored and decides to repeatedly write the digits of 2023 periodically until his number is divisible by 2023^2 (the first 5 versions of his number are $2, 20, 202, 2023, 20232$). How many digits is his final number?

39. Consider $\triangle ABC$ with $\angle A = 90^\circ$ and $AB = 10$. Let D be a point on $\overline{AB}$ such that $BD = 6$. Suppose that the angle bisector of $\angle C$ is tangent to the circle with diameter $\overline{BD}$ and say it intersects $\overline{AB}$ at point E. Given that BE can be expressed as $\frac{a+b\sqrt{c}}{d}$ for positive integers a, b, c, d with $\gcd(a, d) = \gcd(b, d) = 1$ and c square-free, compute $1000a + 100b + 10c + d$.

40. How many ways are there for a rook to return to its original square on a 8×8 chessboard in 6 moves if it starts on a corner? A "move" counts as shifting the rook by a positive number of squares on the board along a row or column. Note that the rook may return back to its original square during an intermediate step within its 6-move path.

41. Consider the following equation:

$$\left(\frac{3}{8}x^3 + 1\right)^3 = \left(\frac{8}{3}(x^2 - 1)\right)^2.$$

This has two real solutions that are greater than 1. One of them is $x = 2$. The other one is $x = \frac{a+\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$. Compute $100a + 10b + c$.

42. Consider polynomial $P(n) = n^3 - 8n^2 + 4n + 12$ with real roots a, b, c. Let Q denote a monic cubic polynomial with roots x, y, z with

$$P(x) + Q(a) = P(y) + Q(b) = P(z) + Q(c) = 0,$$

and P, Q having no roots in common. Given that $x + y + z = 6$ and $x^3 + y^3 + z^3 = 69$, compute $Q(0)$.

43. The integers from 1 to 2022 are placed around a circle in this order. Bob the beaver sits at 1 on the circle. Each minute, he moves left or right to the next number with equal probability. What is the expected number of minutes needed for Bob to visit each number at least once?

44. Andrew and Tiger and playing a game. They start with the number $n^2 + 1$ for some positive integer n and take turns subtracting nonzero perfect squares from the number such that the number is always nonnegative. The player that reduces the number to 0 wins. What is the minimum starting number such that Andrew wins given that he goes first and both players play optimally?

45. Consider $\triangle ABC$ with M denoting the midpoint of $\overline{BC}$. Let the B and C angle bisectors intersect the circle centered at M passing through B, C a second time at U, V, respectively. Let $P = BV \cap CU$, and let the circumcircle of $\triangle UVM$ intersect $\overline{BC}$ a second time at D. Given that $BD = 10$, $CD = 21$, and $\angle A = 60^\circ$, find PD^2.

46. A function $f : \{0, 1, 2, ..., 2021\} \to \{0, 1, 2, ..., 2021\}$ is *strong* if it satisfies

$$f(x + y) \equiv f(x) f(y) \pmod{2022}$$

whenever f is defined on x, y, $x + y$. Find the number of strong functions.

47. The *score* of a nonnegative integer n is defined as $\sum_{i=0}^{9} i f_i(n)^2$, where $f_i(n)$ denotes the number of times the digit i appears in the decimal representation of n. For example, the score of 12341 is $1 \cdot 2^2 + 2 \cdot 1^2 + 3 \cdot 1^2 + 4 \cdot 1^2 = 15$. Compute the floor of the average score of all integers from 0 to $10^{2022} - 1$.

48. In (non-degenerate) triangle ABC, points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $DE \parallel BC$. Let CD and BE intersect $\odot(ABC)$ a second time at F and G, respectively. Let the circumcircles of $\triangle ADF$ and $\triangle AEG$ intersect a second time at Q, and let AQ intersect BC at P. Suppose that $AD = 6$, $BD = 2$, and $AE = 9$. Given that the length of CD is an integer and is minimized, AP^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

49. In a small country with 21 cities, a set of one-way roads (currently not directed) is constructed such that each road starts at one city and ends at a different city with no more than 1 road between 2 cities. Each city belongs to 1 of k political parties, and cities that are connected by a road cannot belong to the same political party. Each city has exactly 18 roads to other cities. Let an *orientation* of these roads be adding a direction to each road. Let an *oriented path* be a path from a city A to another city B that consists of roads all oriented from the city closer to A to the city closer to B, with no city in the path being visited twice. Let the *length* of this path be the number of roads in it. Let the oriented path with maximal length in an orientation be P. What is the minimum length of P over all possible orientations of the roads in this country?

50. In $\triangle ABC$, we have $AB = 5$, $BC = 6$, and $CA = 7$. A variable point P lies on arc BC not containing A of the circumcircle of $\triangle ABC$. Let D and E be the incenters of ABP and ACP, respectively. As P varies on arc BC, the midpoint of $\overline{DE}$ always lies on a fixed circle with area $\frac{m}{n}\pi$, where m, n are coprime, positive integers. Compute $100m + n$.