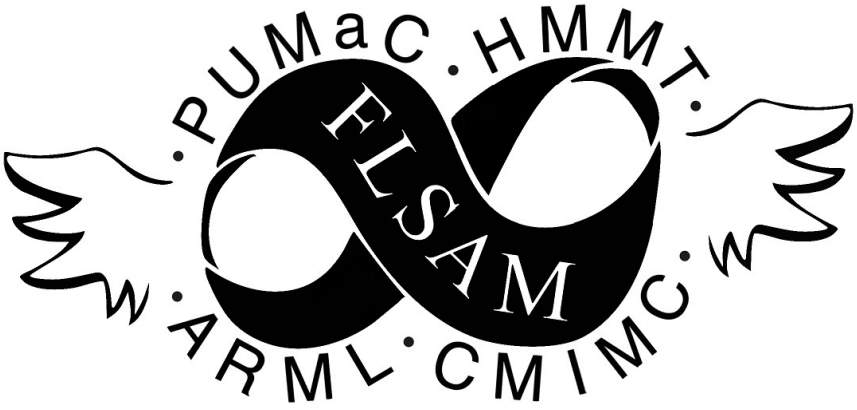
Florida Student Association of Mathematics



2025 ARML Tryout

May 2025

Solutions

Answers

1. 19
2. 329
3. 308
4. 4085
5. 7
6. 4048
7. 129150
8. 14763
9. 36
10. $\frac{1}{12}$
11. 728
12. $\frac{1890}{73}$
13. 310
14. 96

1. A triangle with side lengths 9, 10, and x has area $4x$. Compute the product of all possible values of x.

Proposed by Jacob Khohayting

Answer: 19

Let ABC be a triangle such that $AB = 9$, $BC = M$, and $CA = 10$. Let D be the foot of the perpendicular from A to BC, then $AD = 8$. By Pythagorean Theorem, $DC = 6$ and $BD = \sqrt{17}$. Thus, the possible options for M include $CD - BD$ or $CD + BD$. So,

$$(CD - BD)(CD + BD) = CD^2 - BD^2 = \boxed{19}.$$

2. Let S, A, and M be digits such that

$$17 \cdot 27 \cdot 37 \cdot 47 \cdot 57 \cdot 67 = 3,048,SAM,619.$$

Find $100S + 10A + M$.

Proposed by Jacob Khohayting

Answer: 329

Notice, the number is divisible by $27 \cdot 37 = 999$. Therefore,

$$3 + 048 + \overline{SAM} + 619 \equiv 670 + \overline{SAM} \equiv 0 \mod 999.$$

Thus, $\overline{SAM} = \boxed{329}$.

3. Compute the number of 6-digit positive integers $\overline{ABCDEF}$ satisfying:

- $\overline{AB}$ is divisible by 4
- $\overline{ABCD}$ is divisible by 16
- $\overline{ABCDEF}$ is divisible by 64

Proposed by Jacob Khohayting

Answer: 308

We take $X = \overline{AB}$, $Y = \overline{CD}$, and $Z = \overline{EF}$. The following conditions must be met:

- X must be divisible by 4.
- $10^2 \cdot X + Y$ must be divisible by 16.
- $10^4 \cdot X + 10^2 \cdot Y + Z$ must be divisible by 64.

Since $10^2 \cdot X$ is already divisible by 16, Y must be divisible by 16. Similarly, since $10^4 \cdot X + 10^2 \cdot Y$ is divisible by 64, Z must be divisible by 64. The possible values for X are $12, 16, \ldots, 96$, giving us 22 options. The possible values for Y are $00, 16, 32, \ldots, 96$, providing 7 options. For Z, the possible values are 00 and 64, resulting in 2 options. Therefore, the total number of possible integers is $22 \cdot 7 \cdot 2 = \boxed{308}$.

4. Let $f(k, n)$ denote the number of positive integers less than n that can be represented as the sum of k (not necessarily distinct) powers of 2. Compute

$$f(4, 4096) + f(7, 4096)$$

Proposed by Darsh Patel

Answer: 4085

Consider $f(k, 2^n)$ for any k, n. Any positive integer $a < 2^n$ can be represented as a sum of powers of 2 with a minimum of $s_2(a)$, the sum of the binary digits of a, powers of 2. Starting from this representation, we can repeatedly split 2^i for $i > 0$ into 2 copies of 2^{i-1} until we have a sum of a copies of 2^0. Then a can be represented a sum of anywhere between $s_2(a)$ and a inclusive powers of 2. So, we count a if and only if $s_2(a) \leq k$ and $a \geq k$. The number of $a < 2^n$ with $s_2(a) = j$ is $\binom{n}{j}$, so the first condition alone yields a total of

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$$

Then we subtract k to account for the second condition to get

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} - k$$

Then

$$\begin{aligned}
&f(4, 2^{12}) + f(7, 2^{12}) \\
=&\left(\binom{12}{0} + \binom{12}{1} + \cdots + \binom{12}{4} - 4\right) + \left(\binom{12}{0} + \binom{12}{1} + \cdots + \binom{12}{7} - 7\right) \\
=&\binom{12}{0} + \binom{12}{1} + \cdots + \binom{12}{12} - 11 \\
=&2^{12} - 11 \\
=&\boxed{4085}
\end{aligned}$$

5. A random permutation of the divisors of 900 is chosen. Compute the expected number of pairs of consecutive elements such that the first element divides the second.

Proposed by Aaryan Vaishya

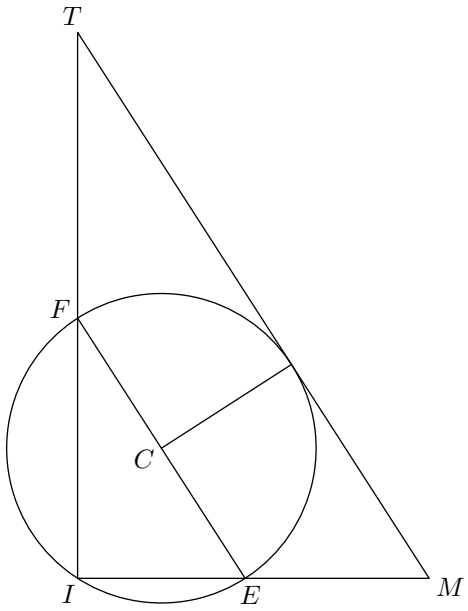
Answer: 7

For any given pair of indices, the probability that those indices result in such a pair is the sum over all divisors m of 900 of the right element being d and the left element being a divisor of m. The former probability is always $\frac{1}{27}$, the latter probability is the number of proper divisors of m divided by 26. Let $d(n)$ denote the number of divisors of n, then d is multiplicative over prime powers, so $\sum_{n|900} d(n) - 1 = (d(1) + d(2) + d(4))(d(1) + d(3) + d(9))(d(1) + d(5) + d(25)) - 27 = 216 - 27 = 189$. Thus the desired probability is $\frac{189}{26 \cdot 27} = \frac{7}{26}$, since there are 26 pairs of consecutive elements, we get the result as $\boxed{7}$.

6. Consider right triangle TIM with $\angle I = 90^\circ$. Circle ω passes through I and is tangent to TM. Let ω intersect IM and TI again at E and F, respectively. Given that EF is parallel to TM and $\frac{TM}{EF} = 2025$, compute the sum of the possible values of $\frac{TI}{IM}$.

Proposed by Tiger Ren

Answer: 4048



Let $a = |TI|, b = |IM|, c = |TM|$. Let C be the midpoint of EF. Then the distance from C to TM is

$$\left(\frac{EM}{IM}\right)h = \left(1 - \frac{1}{2025}\right)h = \left(1 - \frac{1}{2025}\right)\left(\frac{ab}{c}\right)$$

where h is the length of the I-altitude. Also,

$$|CE| = \frac{|EF|}{2} = \left(\frac{FE}{TM}\right)\left(\frac{c}{2}\right) = \frac{1}{2 \cdot 2025}c$$

Then

$$\begin{aligned} &\left(1 - \frac{1}{2025}\right)\left(\frac{ab}{c}\right) = \frac{1}{2 \cdot 2025}c \\ \iff &4048ab = c^2 \\ \iff &a^2 - 4048ab + b^2 = 0 \end{aligned}$$

Then the sum of the possible values of $\frac{a}{b}$ is $\boxed{4048}$.

7. On day n, Jayden, Sihwan, and Tiger each ate x_n, y_n, and z_n tacos, respectively, where $1 \leq x_n < y_n < z_n \leq 20$. After 90 days, they found out that they have eaten in such a way that x_n, y_n, z_n formed an arithmetic sequence every single day; furthermore, they realize they've eaten every possible triple (x, y, z) such that x, y, z are in arithmetic progression exactly once. Compute

$$\sum_{n=1}^{90} x_n y_n z_n.$$

Proposed by Tiger Ren

Answer: 129150

We will write (x_n, y_n, z_n) as $(a - d, a, a + d)$ for some $1 \leq a \leq 20$ and $1 \leq d \leq \min\{a - 1, 20 - a\}$. We remark that we can easily verify that there are indeed

$$1 + 2 + \cdots + 9 + 9 + 8 + \cdots + 1 = 2 \cdot \frac{9 \cdot 10}{2} = 90$$

such triples. The desired sum is

$$\begin{aligned}
&\sum_{\substack{1\le a\le 20\\ 1\le d\le \min\{a-1,20-a\}}} a(a-d)(a+d)\\
=&\sum_{\substack{1\le a\le 20\\ 1\le d\le \min\{a-1,20-a\}}} (a^3-ad^2)\\
=&\left(\sum_{\substack{1\le a\le 20\\ 1\le d\le \min\{a-1,20-a\}}} a^3\right)-\left(\sum_{\substack{1\le d\le 10\\ d+1\le a\le 20-d}} ad^2\right)\\
=&\left(\sum_{a=1}^{10} a^3(a-1)\right)+\left(\sum_{a=11}^{20} a^3(20-a)\right)-\left(\sum_{d=1}^{10} 21d^2(10-d)\right)\\
=&\sum_{n=1}^{10}(n^3(n-1)+(n+10)^3(10-n)-21n^2(10-n))\\
=&\sum_{n=1}^{10}(-210n^2+2000n+10000)\\
=&-210(385)+2000(55)+10000(10)\\
=&\boxed{129150}
\end{aligned}$$

8. Allen is opening a Temu factory with three assembly lines A, B, and C. Find the number of ways to assign his 10 distinguishable workers to the three lines such that within each assembly line the workers can pair up to check each other's work, with no worker left unpaired. Note that an assembly line may contain no workers.

Proposed by Yusuf Sheikh

Answer: 14763

We use generating functions. The choice of assigning each worker to one of A, B, C is represented by the polynomial $x+y+z$, where x, y, z correspond to A, B, C respectively. Then the choices for assigning all 10 workers is represented by

$$(x+y+z)^{10}$$

where the coefficient of $x^iy^jz^k$ is the number of ways to assign workers in each A, B, C have i, j, k workers respectively. We want to count the cases in which i, j, k are all even. By the Roots of Unity Filter, this is

$$\frac{1}{2^3}\sum_{a,b,c\in\{-1,1\}}(a+b+c)^{10}=\frac{3^{10}+3}{4}=\boxed{14763}$$

9. Suppose a, b, and c are real numbers such that

$$ab+bc+ca=40-a^2=45-b^2=72-c^2.$$

Compute $|abc|$.

Proposed by Jacob Khohayting

Answer: 36

Notice, we have the following three equations:

$$\begin{aligned}
a^2+ab+bc+ca&=(a+b)(a+c)=40\\
b^2+ab+bc+ca&=(b+a)(b+c)=45\\
c^2+ab+bc+ca&=(c+a)(c+b)=72.
\end{aligned}$$

Then, $(a+b)(b+c)(c+a) = \sqrt{(a+b)^2(b+c)^2(c+a)^2} = \pm 360$. Hence, dividing each of the above equations yields:

$$b+c = \pm 9 \qquad a+c = \pm 8 \qquad a+b = \pm 5.$$

Then, $a+b+c = \dfrac{(a+b)+(b+c)+(c+a)}{2} = \pm 11$. Therefore, $(a,b,c) = (\pm 2, \pm 3, \pm 6)$ and $|abc| = \boxed{36}$.

10. Andrew is at the center of equilateral triangle P_0 with side length 1. At step n, he divides P_{n-1} into 4 congruent equilateral triangles and picks one of them at random to be P_n. Let C_n be the center of P_n. Compute the least real number r such that for all n, the expected value of $C_0C_n^2$ is at most r.

Proposed by Aaryan Vaishya

Answer: $\frac{1}{12}$

We will represent the position of C_n using complex numbers; C_0 is represented by 0. Let $\omega = \operatorname{cis}\left(\frac{\pi}{3}\right)$. Note that the distance C_0C_1 is either 0 or $\frac{\sqrt{3}}{6}$. Let a_n be the number of times 0 appears among $\overrightarrow{C_0C_1}, \overrightarrow{C_1C_2}, \ldots, \overrightarrow{C_{n-2}C_{n-1}}$. Then we can represent $\overrightarrow{C_{n-1}C_n}$ as $\frac{\sqrt{3}}{6}\left(\frac{\omega^{a_n}v_n}{2^{n-1}}\right)$, where v_n is randomly selected from $\{0, 1, \omega^2, \omega^4\}$. Then C_n is represented by

$$\frac{\sqrt{3}}{6}\left(\omega^{a_1}v_1 + \frac{\omega^{a_2}v_2}{2} + \cdots + \frac{\omega^{a_n}v_n}{2^{n-1}}\right)$$

Then

$$\begin{aligned}
&\mathbb{E}[C_0C_n^2] \\
=&\mathbb{E}\left[\frac{1}{12}\left(\omega^{a_1}v_1 + \frac{\omega^{a_2}v_2}{2} + \cdots + \frac{\omega^{a_n}v_n}{2^{n-1}}\right)\overline{\left(\omega^{a_1}v_1 + \frac{\omega^{a_2}v_2}{2} + \cdots + \frac{\omega^{a_n}v_n}{2^{n-1}}\right)}\right] \\
=&\frac{1}{12}\mathbb{E}\left[\left(\omega^{a_1}v_1 + \frac{\omega^{a_2}v_2}{2} + \cdots + \frac{\omega^{a_n}v_n}{2^{n-1}}\right)\left(\omega^{-a_1}\overline{v_1} + \frac{\omega^{-a_2}\overline{v_2}}{2} + \cdots + \frac{\omega^{-a_n}\overline{v_n}}{2^{n-1}}\right)\right] \\
=&\frac{1}{12}\mathbb{E}\left[\left(\sum_i \frac{v_i\overline{v_1}}{4^i}\right) + \left(\sum_{i\neq j} \frac{\omega^{a_i-a_j}v_i\overline{v_j}}{2^{i+j}}\right)\right] \\
=&\frac{1}{12}\left(\left(\sum_i \frac{\mathbb{E}(v_i\overline{v_i})}{4^i}\right) + \left(\sum_{i\neq j} \frac{\mathbb{E}(\omega^{a_i-a_j}v_i\overline{v_j})}{2^{i+j}}\right)\right)
\end{aligned}$$

Now, $\mathbb{E}(v_i\overline{v_i})$ is $\frac{1}{4}(0+1+1+1) = \frac{3}{4}$, and $\mathbb{E}(\omega^{a_i-a_j}v_i\overline{v_j})$ is $\mathbb{E}(\omega^{a_i-a_j}v_i)\mathbb{E}(\overline{v_j}) = \mathbb{E}(\omega^{a_i-a_j}v_i)\cdot 0 = 0$ if $i<j$ or $\mathbb{E}(\omega^{a_i-a_j}\overline{v_j})\mathbb{E}(v_i) = \mathbb{E}(\omega^{a_i-a_j}\overline{v_j})\cdot 0 = 0$ if $i>j$. Then this sum is just

$$\left(\frac{1}{12}\right)\left(\frac{3}{4}\right)\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-1}}\right)$$

Then

$$r = \left(\frac{1}{12}\right)\left(\frac{3}{4}\right)\left(1 + \frac{1}{4} + \cdots\right) = \left(\frac{1}{12}\right)\left(\frac{3}{4}\right)\left(\frac{4}{3}\right) = \boxed{\frac{1}{12}}$$

11. Compute the sum of all distinct prime factors of 819604.

Proposed by Aaryan Vaishya

Answer: 728

We have

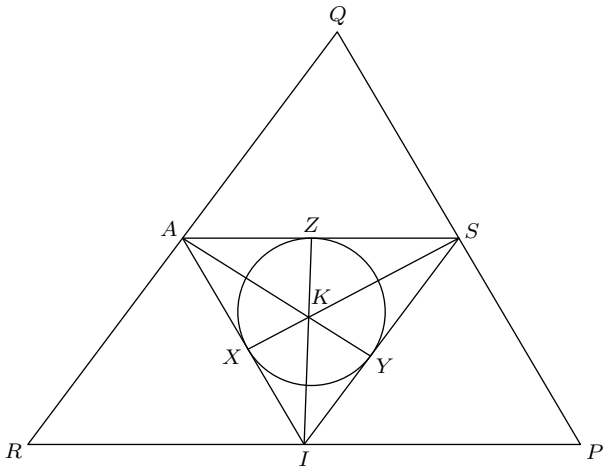
$$\begin{aligned}
&819604\\
=&30^4+98^2\\
=&30^4+4\cdot 7^4\\
=&(30^2+2\cdot 7^2+2\cdot 7\cdot 30)(30^2+2\cdot 7^2-2\cdot 7\cdot 30)\\
=&(578)(1418)\\
=&2^2\cdot 17^2\cdot 709
\end{aligned}$$

Upon manually checking that 709 is prime, we get the answer as $2+17+709=\boxed{728}$

12. Triangle $\triangle PQR$ has $PQ=13$, $QR=14$, $RP=15$. Let the midpoints of sides PQ, QR, RP be S, A, I. The incircle of triangle $\triangle SAI$ touches its sides AI, IS, SA at X, Y, Z; SX, AY, and IZ all intersect at a common point K. Find the area of $\triangle KRP$.

Proposed by Yusuf Sheikh

Answer: $\frac{1890}{73}$



We use barycentric coordinates with respect to triangle ISA. We have

$$\begin{aligned}
Q&=(1,1,-1)\\
R&=(1,-1,1)\\
P&=(-1,1,1)\\
Z&=(0:s-c:s-b)\\
X&=(s-c:0:s-a)\\
Y&=(s-b:s-a:0)\\
K&=((s-b)(s-c):(s-c)(s-a):(s-a)(s-b))
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{[KRP]}{[ISA]} \\
=&\frac{1}{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)}\begin{vmatrix}(s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b)\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix} \\
=&\frac{-2((s-a)(s-b)+(s-a)(s-c))}{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)} \\
=&\frac{-2a(s-a)}{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)} \\
=&\frac{-2(15)(6)}{(6)(7)+(7)(8)+(8)(6)} \\
=&-\frac{90}{73}
\end{aligned}$$

Then

$$|[KRP]| = \frac{90}{73}\cdot[ISA] = \frac{90}{73}\cdot\frac{1}{4}\cdot[QRP] = \frac{90}{73}\cdot\frac{1}{4}\cdot 84 = \boxed{\frac{1890}{73}}$$

13. Compute the number of ordered pairs (a, b) of elements from the set $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{100}\}$ such that ab is an integer.

Proposed by Aaryan Vaishya

Answer: 310

We can represent every positive integer n in the form kd^2, where k and d are positive integers such that k is squarefree; call k the squarefree part of n. We then see that mn is a perfect square if and only if m and n have the same squarefree part. Now, the number of positive integers at most 100 that have squarefree part k for some squarefree k is

$$\left\lfloor \sqrt{\frac{100}{k}} \right\rfloor$$

We then wish to compute

$$\sum_{k \text{ squarefree}} \left\lfloor \sqrt{\frac{100}{k}} \right\rfloor^2$$

Letting $f(x)$ be the largest perfect square at most x, this is

$$\sum_{k \text{ squarefree}} f\left(\frac{100}{k}\right)$$

Now, we have that $f\left(\frac{100}{k}\right)$ is 0 for $k > 100$ and 1 for $25 < k \leq 100$. We then have

$$\sum_{k \text{ squarefree}} f\left(\frac{100}{k}\right) = \left(\sum_{\substack{k \text{ squarefree}\\ k\leq 25}} \left(f\left(\frac{100}{k}\right) - 1\right)\right) + \left(\sum_{\substack{k \text{ squarefree}\\ k\leq 100}} 1\right)$$

The first sum can be computed directly as

$$99\cdot|\{1\}| + 48\cdot|\{2\}| + 24\cdot|\{3\}| + 15\cdot|\{5,6\}| + 8\cdot|\{7,10,11\}| + 3\cdot|\{13,14,15,17,19,21,22,23\}| = 249$$

The second sum, the number of squarefree numbers at most 100, is, by the Principle of Inclusion-Exclusion,

$$\sum_{k \text{ squarefree}} \mu(k) \left\lfloor \frac{100}{k^2} \right\rfloor$$

where μ is the Möbius function, which we compute as

$$\left\lfloor \frac{100}{1^2} \right\rfloor - \left\lfloor \frac{100}{2^2} \right\rfloor - \left\lfloor \frac{100}{3^2} \right\rfloor - \left\lfloor \frac{100}{5^2} \right\rfloor + \left\lfloor \frac{100}{6^2} \right\rfloor - \left\lfloor \frac{100}{7^2} \right\rfloor + \left\lfloor \frac{100}{10^2} \right\rfloor = 61$$

Then the final count is $249 + 61 = \boxed{310}$.

14. For a positive integer n, define

$$f(n) = \left\lfloor \sum_{i=1}^{n} \gcd(i, n) \cos\left(\frac{2\pi i}{n}\right) \right\rfloor$$

Compute $f(f(f(2025)))$.

Proposed by Darsh Patel

Answer: 96

Let $\omega = \operatorname{cis}\left(\frac{2\pi}{n}\right)$. Then

$$\sum_{i=1}^{n} \gcd(i, n) \cos\left(\frac{2\pi i}{n}\right) = \Re\left(\sum_{i=1}^{n} \gcd(i, n)\omega^i\right)$$

We have

$$\sum_{i=1}^{n} \gcd(i, n)\omega^i = \sum_{d|n} \sum_{\gcd(i,n)=d} d\omega^i = \sum_{d|n} \sum_{\gcd\left(i, \frac{n}{d}\right)=1} d\omega^{di}$$

Now note that

$$\sum_{\gcd\left(i, \frac{n}{d}\right)=1} \omega^{di}$$

is the sum of the primitive $\frac{n}{d}$th roots of unity, which is $\mu\left(\frac{n}{d}\right)$, where μ is the Möbius function. Then

$$\sum_{d|n} \sum_{\gcd\left(i, \frac{n}{d}\right)=1} d\omega^{di} = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = (I * \mu)(n)$$

where I is the identity function and $*$ denotes Dirichlet convolution. We now note that

$$I * \mu = \varphi \iff \varphi * 1 = I$$

So,

$$f(n) = \lfloor \Re(\varphi(n)) \rfloor = \varphi(n)$$

We then compute

$$\varphi(\varphi(\varphi(3^4 \cdot 5^2))) = 2^5 \cdot 3 = \boxed{96}$$