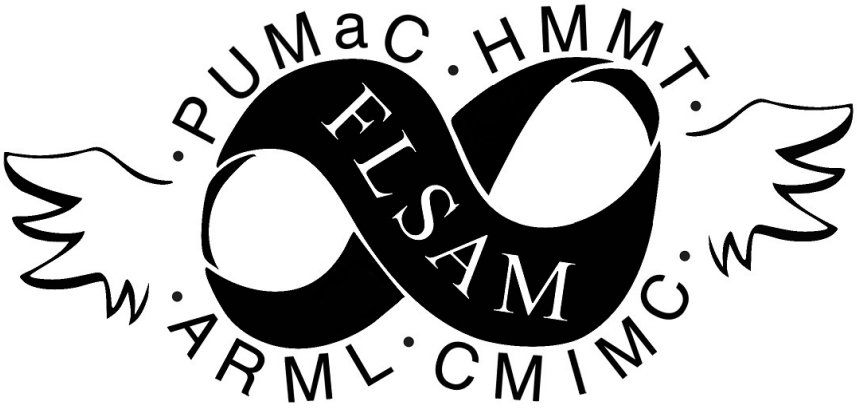
Florida Student Association of Mathematics



2024-2025 Florida Online Math Open

Winter 2024-2025

Rules

1. You will have 7 days to complete the contest. The test will be released at 7 PM on December 28 and submissions will be due by 7 PM on January 4.
2. This is a team event with teams of up to 6 students; collaboration with your teammates is encouraged. Discussion about the contest or its problems is not allowed between members of different teams or anyone not competing.
3. Each correct answer is worth 1 point. Incorrect or omitted answers are both worth 0 points.
4. Ties will be broken by looking at the most difficult question each team solved. If there is still a tie, the process will move on to the second most difficult question solved and so on. Difficulty is determined by the number of teams that solved a question, with fewer solves meaning a more difficult question.
5. Four function calculators are permitted, but graphing calculators and electronic programs are not. The only drawing aids that are permitted are graph paper, rulers, compasses, and protractors.
6. All answers are nonnegative integers. Any non-integer final value will be converted to a nonnegative integer.
7. Updates, including clarifications and errata, will be posted in the `#flomo-updates` channel of the Discord. Disputes that are answered by such updates will not be accepted. If you wish to ask a clarification question, contact either Aaryan Vaishya or Darsh Patel privately through Discord.

1. Compute the sum of the non-real roots of $2x^{2025} + 0x^2 + 2x + 4 = 0$.

2. In triangle WUO, points S and N are the midpoints of sides WU and UO, respectively. Suppose the perimeter of triangle SUN and quadrilateral $SNOW$ are 20 and 24, respectively. Compute $WU + UO$.

3. Daniel randomly and uniformly chooses a real number n in the interval $[0, 1]$. The expected value of

$$\lfloor 2n \rfloor + \lfloor 3n \rfloor + \lfloor 5n \rfloor$$

can be expressed as $\frac{p}{q}$ for some relatively prime positive integers p and q. Compute $100p + q$.

4. Andrew chooses 30 sets of real numbers, each of which is the union of some 20 (not necessarily disjoint) closed intervals. Let S be the intersection of all the sets, and let x be the smallest integer such that S can be represented as the union of x closed intervals. Over all possible choices Andrew could have made, compute the largest possible value of x.

5. Compute the sum of the positive integers T for which there exist unique positive integers X and Y such that

$$\frac{2}{71} < \frac{X}{T} < \frac{Y}{T} < \frac{3}{14}$$

6. Over reals x and y satisfying $x^2 + y^2 \leq 1$, the maximum possible value of $x^2 + xy - y^2$ can be expressed as $\frac{\sqrt{a}}{b}$ for some positive integers a and b with a squarefree. Compute $100a + b$.

7. In rhombus $ABCD$, M and N are points on line segments AB and CD respectively such that $AMCN$ is a parallelogram with area 240. If $AC = 30$ and $MN = 20$, compute the area of $ABCD$.

8. Compute the sum of all real numbers p in the interval $[1, 69]$ such that

$$1, \{p\}, \{p^2\}$$

forms a geometric sequence with nonzero ratio, in that order.

[For all reals x, we define $\{x\} = x - \lfloor x \rfloor$.]

9. Let $n \geq 25$ be a positive integer. In regular polygon $A_1A_2 \cdots A_n$, define X as the reflection of A_1 across A_3. Given that X, A_5, and A_{25} are collinear, find n.

10. Johnny and Allen have magical dice that can roll any positive integer. Johnny's dice rolls any n with probability $\frac{1}{2^n}$, and Allen's dice rolls any $n \neq 2$ with probability $\frac{1}{3^n}$. They roll until one of them rolls higher than the other, with the one doing so being the winner. The probability Allen wins can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

11. Compute the product of all possible values of L such that the polynomial $x^4 + 6x^3 + Lx^2 + 12x - 1$ can be factored into two non-constant polynomials with integer coefficients.

12. The number of (possibly empty) subsets of the set $\{1, 2, \ldots, 101\}$ such that no two elements differ by exactly 10 can be expressed as $p^a q^b r^c$ for some distinct primes p, q, r and positive integers a, b, c. Compute $(p + q + r)(a + b + c)$.

13. Cyclic hexagon $ABCDEF$, whose vertices lie in that order, is such that

$$\angle A, \angle B, \angle D, x, \angle C, \angle E, \angle F$$

forms an arithmetic sequence for some x. Also, $AE = 7$ and $BF = 8$. Compute the square of the circumradius of $ABCDEF$.

14. Compute the number of positive integers $b \geq 2$ such that 81! has more trailing zeroes than 80! when both are written in base b.

15. A right trapezoid $ABCD$ with bases AB and CD and side BC perpendicular to its bases is inscribed in an isosceles right triangle so that A and D lie on the hypotenuse and B and C lie on different legs. Given that $AB = 30$, $BC = 40$, and $CD = 10$, compute the area of the isosceles right triangle.

16. Compute the sum of the squares of the side lengths of a non-degenerate, non-self-intersecting, convex 40-gon with smallest possible perimeter, all of whose vertices are lattice points.

17. Reals a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = 1$$
$$ab = cd$$

Compute the greatest possible value of

$$29a + 11b + 61c + 19d$$

18. Consider a positive integer n. In a game of musical chairs, there are n people sitting in chairs $1, 2, \ldots, n$. A permutation p of the chairs is chosen at random. Each player repeatedly travels from their current chair i to chair $p(i)$, taking 2^{-i} seconds to do so, until they end up at their original chair. Let E_n be the expected amount of time that elapses before everyone is done. The least real number r such that $E_n \leq r$ for all $n \geq 2$ can be expressed as $\frac{a}{b}$ for some relative prime positive integers a and b. Compute $100a + b$.

(Note that multiple people can be in a seat at the same time. Also, if $p(i) = i$, then the player starting in seat i is done after 2^{-i} seconds.)

19. Let $O = (0, 0)$ be the origin. Lattice points A, B, and C are such that their squared distances to the origin

$$OA^2, OB^2, OC^2$$

are consecutive integers in that order and are all less than 250. Compute the sum of all distinct possible values of OB^2.

20. Consider triangle ABC and points D, E, and F on sides BC, CA, and AB respectively such that AD, BE, and CF concur. Let lines k and l parallel to AB through E and D respectively intersect CF at X and Z respectively. Let line k intersect segment FD at Q and let lines CF and ED intersect at Y. If $\frac{FQ}{DQ} = \frac{17}{27}$, then $\frac{XF}{XY}$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

21. A non-degenerate triangle is called *durable* if it has perimeter 1 and there exists a non-degenerate triangle whose side lengths are the lengths of its altitudes. The length of the smallest closed interval that contains each side length of every durable triangle can be expressed as $\frac{\sqrt{a}-b}{c}$ for some positive integers a, b, and c with a squarefree and $\gcd(b, c) = 1$. Compute $10000a + 100b + c$.

22. A bijective function $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ is called *really-even* if there exists a positive integer k such that $f^{2^k}(x) = x$ is satisfied for all $x \in \{1, 2, \ldots, 8\}$. Compute the number of really-even f.

23. Complex numbers x and y satisfy

$$x^3 + y^2 = 8(x + 1)$$
$$y^3 + x^2 = 8(y + 1)$$

Compute the sum of x^2 over all distinct possible values of x.

24. Circles ω_1, ω_2 centered at O_1, O_2 respectively intersect at distinct points A and B. An external tangent of the circles meets ω_1 and ω_2 at E and F respectively such that point A is closer to segment EF. Given that E, A, O_2 are collinear and $EA = 3$, the minimum possible sum of the radii of two circles can be expressed as $\frac{a\sqrt{b}+c}{d}$ for some positive integers a, b, c, and d with b squarefree and $\gcd(a, c, d) = 1$. Compute $1000000a + 10000b + 100c + d$.

25. Compute the area of the finite region bounded by the graph of

$$y = |||\ldots||||x - 0| - 1| - 2| - 3|\ldots| - 199| - 200|$$

and the x-axis.

26. Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the feet of the A, B, and C-altitudes respectively. Let T be the intersection of lines AD and EF. Let P be the orthocenter of BTC. The length PT can be expressed as $\frac{a}{b}$, for some relatively prime positive integers a and b. Compute $100a + b$.

27. Permutations x and y of the word $ABCDE$ are chosen randomly. Let $F(x, y)$ be the length of the shortest string that contains both permutations as substrings. The expected value of $F(x, y)$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

28. Let $\omega = \text{cis}\left(\frac{2\pi}{7}\right)$. A polynomial $P(x)$ with rational coefficients has minimum possible degree while satisfying

$$P(\omega) = \frac{1}{(2-\omega)^2}$$

Then $P(1)$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

29. For any positive integer n and real r, let $f(n, r)$ be the least positive integer k for which there exist relatively prime perfect squares a and b with

- $a + b = n^k$
- $|a - b| \leq r(n^k)$

or 0 if no such k exists. Given n such that $f\left(n, \frac{1}{2}\right) = 2024$, compute $f\left(n, \sqrt{\frac{1}{2}}\right)$.

30. Consider cyclic hexagon $ABDEFC$ with $AB = AC$. Define $P = AD \cap BE$, $Q = CD \cap BF$, $R = AF \cap CE$, $S = AD \cap BF$, and $T = AF \cap CD$. Point K lies on line TS such that $\angle SKQ = \angle ACE$. Given that $AB = 20$, $PQ = 7$, $PR = 18$, and $SK = 5$, TK can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.