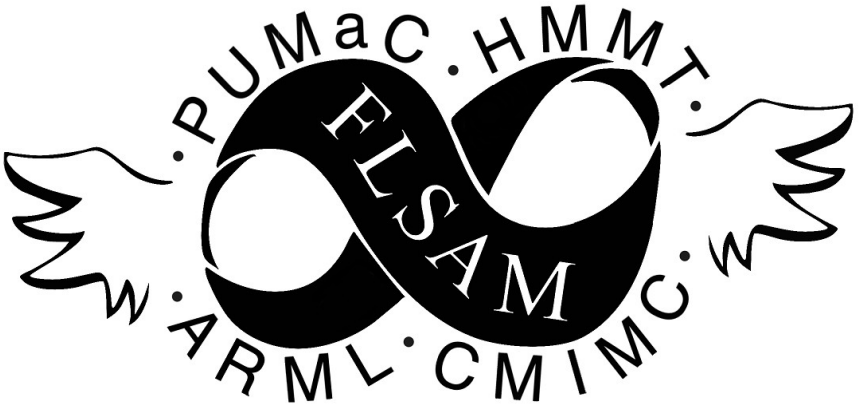
Florida Student Association of Mathematics



2024-2025 Florida Online Math Open

Winter 2024-2025

Solutions

Problem Contributors

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Thank you to everyone who submitted problems!

Answers

1. 1
2. 36
3. 702
4. 571
5. 60
6. 502
7. 400
8. 214234
9. 48
10. 1427
11. 8748
12. 13090
13. 19
14. 12720747185727
15. 1280
16. 372
17. 70
18. 304
19. 559
20. 6127
21. 50104
22. 11264
23. 32
24. 6020302
25. 4030000
26. 12156
27. 113420
28. 1315429
29. 1518
30. 5507

1. Compute the sum of the non-real roots of $2x^{2025} + 0x^2 + 2x + 4 = 0$.

Proposed by Aaryan Vaishya

Answer: 1

Note that $2x^{2025} + 0x^2 + 2x + 4$ is increasing in x and so has at most 1 real root. Then -1 is its only real root. By Vieta Formulas, the sum of the roots is 0, so the sum of the non-real roots is $\boxed{1}$.

2. In triangle WUO, points S and N are the midpoints of sides WU and UO, respectively. Suppose the perimeter of triangle SUN and quadrilateral $SNOW$ are 20 and 24, respectively. Compute $WU + UO$.

Proposed by Jacob Khohayting

Answer: 36

The key is to note that $p(SNOW) - p(\triangle SUN) = 2 \cdot SN = 4$. Hence, $SN = 2$. But,

$$p(\triangle WUO) = p(\triangle SUN) + p(SNOW) - 4 \cdot SN = \boxed{36}.$$

3. Daniel randomly and uniformly chooses a real number n in the interval $[0, 1]$. The expected value of

$$\lfloor 2n \rfloor + \lfloor 3n \rfloor + \lfloor 5n \rfloor$$

can be expressed as $\frac{p}{q}$ for some relatively prime positive integers p and q. Compute $100p + q$.

Proposed by Luis Fonseca

Answer: 702

By Linearity of Expectation,

$$\mathbb{E}[\lfloor 2n \rfloor + \lfloor 3n \rfloor + \lfloor 5n \rfloor] = \mathbb{E}[\lfloor 2n \rfloor] + \mathbb{E}[\lfloor 3n \rfloor] + \mathbb{E}[\lfloor 5n \rfloor] = \frac{1}{2}(0+1) + \frac{1}{3}(0+1+2) + \frac{1}{5}(0+1+2+3+4) = \frac{7}{2}$$

giving $700 + 2 = \boxed{702}$ as desired.

4. Andrew chooses 30 sets of real numbers, each of which is the union of some 20 (not necessarily disjoint) closed intervals. Let S be the intersection of all the sets, and let x be the smallest integer such that S can be represented as the union of x closed intervals. Over all possible choices Andrew could have made, compute the largest possible value of x.

Proposed by Aaryan Vaishya

Answer: 571

Each closed interval formed by the intersection can be assigned to the end of some interval in one of the sets. If multiple such endpoints exist, assign one arbitrarily. Then consider the endpoints of last intervals in each set, at most one can correspond to an interval in the union, so the number of intervals in the union is at most $30 \cdot 20 - (30 - 1) = 571$.

A simple inductive construction can be outlined as follows. Start with one set, we then have 20 intervals in the union. Now for each new set we add, take the first interval in the union and make all the 20 intervals from the set overlap with the interval in the union, and the last interval from the set overlap with all intervals in the union. This adds 19 intervals to the union, so repeating this 29 times gives $29 \cdot 19 + 20 = 571$ intervals in the union.

Since we cannot have more than 571 intervals, and a construction for 571 exists, the answer is $\boxed{571}$.

5. Compute the sum of the positive integers T for which there exist unique positive integers X and Y such that

$$\frac{2}{71} < \frac{X}{T} < \frac{Y}{T} < \frac{3}{14}$$

Proposed by Jacob Khohayting

Answer: 60

We want two such values of V to satisfy:

$$\frac{2}{71} < \frac{V}{T} < \frac{3}{14} \quad \longrightarrow \quad \frac{14}{3}V < T < \frac{71}{2}V.$$

Notice, if $V \geq 3$ satisfied the inequality, then there would be at least three solutions. This can be seen as at least one of the following triplets would contain solutions: $\{V-2, V-1, V\}$, $\{V-1, V, V+1\}$, and $\{V, V+1, V+2\}$. Hence, we must find the possible values of T such that $V = 1$ and $V = 2$ are both solution. Henceforth, T must satisfy:

$$\frac{14}{3} < T < \frac{71}{2} \quad \text{and} \quad \frac{28}{3} < T < 71.$$

In addition, T cannot satisfy the inequality when $V \geq 3$, so the inequality $T > 14$ cannot be satisfied. Thus, $\frac{28}{3} < T \leq 14$. Therefore, the possible values of T include $T = 10, 11, 12, 13$, and 14. Hence, the sum of possible values of T are $\boxed{60}$.

6. Over reals x and y satisfying $x^2 + y^2 \leq 1$, the maximum possible value of $x^2 + xy - y^2$ can be expressed as $\frac{\sqrt{a}}{b}$ for some positive integers a and b with a squarefree. Compute $100a + b$.

Proposed by Jingyi Li

Answer: 502

Let $x = r\cos\theta$ and $y = r\sin\theta$. So the following calculations can be done.

$$\begin{aligned}
&x^2 + xy - y^2 \\
&= r^2\cos^2\theta + r^2\cos\theta\sin\theta - r^2\sin^2\theta \\
&= r^2\left(\cos^2\theta - \sin^2\theta + \cos\theta\sin\theta\right) \\
&= r^2\left(\cos 2\theta + \frac{1}{2}\sin 2\theta\right) \\
&= r^2 \cdot \frac{\sqrt{5}}{2}\sin\left(2\theta + \arctan 2\right)
\end{aligned}$$

Since the maximum value for r is 1, the maximum value for $x^2 + xy - y^2$ is $\frac{\sqrt{5}}{2}$, giving the requested sum as $\boxed{502}$.

7. In rhombus $ABCD$, M and N are points on line segments AB and CD respectively such that $AMCN$ is a parallelogram with area 240. If $AC = 30$ and $MN = 20$, compute the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: 400

Let X be the projection from M to AC, and Y be the center of the kite. By symmetry, M, N, Y are collinear. We see that $[AMC] = 120$, so $\frac{1}{2} \cdot MX \cdot AC = 120$, giving $MX = 8$. Since $MN = 20$, $MY = 10$, so $XY = 6$. Then $AX = 9$, so since $\triangle AXM \sim \triangle AYB$, we have $[ABCD] = 4[AYB] = 4\frac{AY^2}{AX^2}[AXM] = 4\frac{15^2}{9^2}(\frac{1}{2} \cdot 9 \cdot 8) = \boxed{400}$.

8. Compute the sum of all real numbers p in the interval $[1, 69]$ such that

$$1, \{p\}, \{p^2\}$$

forms a geometric sequence with nonzero ratio, in that order.

[For all reals x, we define $\{x\} = x - \lfloor x \rfloor$.]

Proposed by Jayden Lang

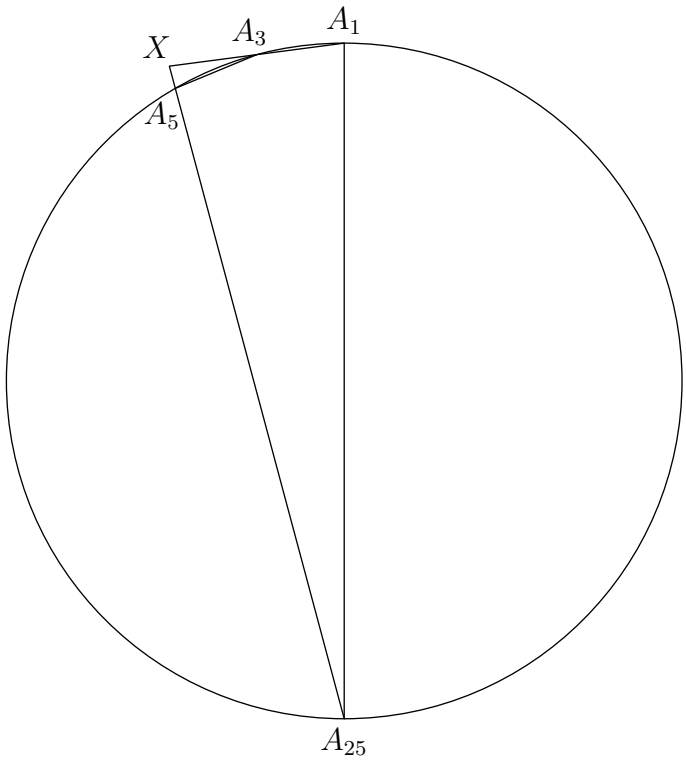
Answer: 214234

Since the three numbers form a geometric sequence, $1 \cdot (p^2 - \lfloor p^2 \rfloor) = (p - \lfloor p \rfloor)^2$. Rearranging and rewriting, $\lfloor p^2 \rfloor + \lfloor p \rfloor^2 = 2p\lfloor p \rfloor$. Since the left side is an integer, the right side must also be an integer and thus p must be in the form $\lfloor p \rfloor + \frac{n}{2\lfloor p \rfloor}$, where $n < 2\lfloor p \rfloor$ is a positive integer. Note that n cannot be zero as then p would be an integer, so the values $p - \lfloor p \rfloor$ and $p^2 - \lfloor p^2 \rfloor$ would both be zero which is a contradiction. Now, substituting in for p on the RHS, $\lfloor p^2 \rfloor + \lfloor p \rfloor^2 = 2\left(\lfloor p \rfloor + \frac{n}{2\lfloor p \rfloor}\right)\lfloor p \rfloor = 2\lfloor p \rfloor^2 + n$. It follows that $\lfloor p^2 \rfloor = \lfloor p \rfloor^2 + n$, and subsituting again for p on the LHS yields $\left\lfloor \left(\lfloor p \rfloor + \frac{n}{2\lfloor p \rfloor}\right)^2 \right\rfloor = \left\lfloor \lfloor p \rfloor^2 + n + \frac{n^2}{4\lfloor p \rfloor^2} \right\rfloor = \lfloor p \rfloor^2 + n$. Since $\lfloor p \rfloor^2 + n$ is always an integer, and $\frac{n^2}{4\lfloor p \rfloor^2}$ is always less than 1, it follows that this equality holds for all $0 < n < 2\lfloor p \rfloor$. So the sum we want to compute is $\sum_{k=1}^{68} \sum_{n=1}^{2k-1} \left(k + \frac{n}{2k}\right) = \sum_{k=1}^{68} \sum_{n=1}^{2k-1} k + \sum_{k=1}^{68} \frac{\sum_{n=1}^{2k-1} n}{2k} = \sum_{k=1}^{68} k\,(2k-1) + \sum_{k=1}^{68} \frac{\frac{(2k-1)(2k)}{2}}{2k} = \sum_{k=1}^{68} \left(k\,(2k-1) + k - \frac{1}{2}\right) = \sum_{k=1}^{68} \left(2k^2 - \frac{1}{2}\right) = 2\sum_{k=1}^{68} k^2 - 34 = \frac{2(68)(68+1)(2\cdot 68+1)}{6} - 34 = \boxed{214234}$.

9. Let $n \geq 25$ be a positive integer. In regular polygon $A_1A_2 \cdots A_n$, define X as the reflection of A_1 across A_3. Given that X, A_5, and A_{25} are collinear, find n.

Proposed by Jacob Khohayting

Answer: 48



Notice, $\triangle XA_3A_5 \sim \triangle XA_{25}A_1$. Thus,

$$\angle XA_3A_5 = \angle XA_{25}A_1 = \angle A_5A_{25}A_1 = 180 \cdot \frac{4}{n}, \text{ and}$$
$$\angle XA_5A_3 = \angle XA_1A_{25} = \angle A_5A_1A_{25} = 180 \cdot \frac{22}{n}.$$

Hence, $\angle A_3XA_5 = \left(1 - \frac{26}{n}\right) \cdot 180$. In addition, $A_3A_5 = A_1A_3 = XA_3$, so $\angle XA_5A_3 = \angle A_5XA_3$. Therefore, $1 - \frac{26}{n} = \frac{22}{n}$ and $n = \boxed{48}$.

10. Johnny and Allen have magical dice that can roll any positive integer. Johnny's dice rolls any n with probability $\frac{1}{2^n}$, and Allen's dice rolls any $n \neq 2$ with probability $\frac{1}{3^n}$. They roll until one of them rolls higher than the other, with the one doing so being the winner. The probability Allen wins can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Roger He

Answer: 1427

We first solve for the probability that Johnny wins on a single roll. This is equivalent to $\sum_{i=1}^{\infty}(\frac{1}{3^i}\sum_{k=i+1}^{\infty}\frac{1}{2^k}) + \frac{1}{2}\sum_{i=3}^{\infty}\frac{1}{2^i}$.

We have $\sum_{i=n}^{\infty}\frac{1}{2^i} = \frac{1}{2^{n-1}}$. Using this in our original equation, we evaluate it to be $\sum_{i=1}^{\infty}\frac{1}{6^i} + \frac{1}{2^3}$. Using the geometric series formula we get $\sum_{i=1}^{\infty}\frac{1}{6^i} + \frac{1}{2^3} = \frac{1}{5} + \frac{1}{8} = \frac{13}{40}$. The probability that they score the same is $(\sum_{i=1}^{\infty}\frac{1}{2^i}\cdot\frac{1}{3^i}) + \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{5} + \frac{1}{8} = \frac{13}{40}$.

The probability Allen wins is equal to the probability that on a roll where one player rolls higher, Allen rolls higher. This is $1 - \frac{\frac{13}{40}}{1-\frac{13}{40}} = \frac{14}{27}$. The desired value is then 1427.

11. Compute the product of all possible values of L such that the polynomial $x^4 + 6x^3 + Lx^2 + 12x - 1$ can be factored into two non-constant polynomials with integer coefficients.

Proposed by Jacob Khohayting

Answer: 8748

Denote the given polynomial as $P(x)$. Then, $P(x) = A(x)B(x)$. We have two cases:

- $\deg(A) = 1$ or $\deg(B) = 1$. Suppose $\deg(A) = 1$, then $A(x) = x \pm 1$. Firstly, if $A(x) = x + 1$, then:
$$P(-1) = (-1)^4 + 8(-1)^3 + L(-1)^2 + 16(-1) - 1 = 1 - 6 + L - 12 - 1 = 0.$$
Thus, $L = 18$. Lastly, $A(x) = x - 1$, then:
$$P(1) = 1^4 + 6 \cdot 1^3 + L \cdot 1^2 + 12 \cdot 1 - 1 = 1 + 6 + L + 12 - 1 = 0.$$
Thus, $L = -18$.
- $\deg(A) = \deg(B) = 2$. Then,
$$\begin{aligned} x^4 + 6x^3 + Lx^2 + 12x - 1 = A(x)B(x) &= \left(x^2 + ax + 1\right)\left(x^2 + bx - 1\right) \\ &= x^4 + (a+b)x^3 + (ab)x^2 + (b-a)x - 1. \end{aligned}$$
Thus, $a + b = 6$ and $b - a = 12$ which solves $(a, b) = (-3, 9)$, thus $ab = -27$.

Thus, the product of all possible values is $18 \cdot (-18) \cdot (-27) = \boxed{8748}$.

12. The number of (possibly empty) subsets of the set $\{1, 2, \ldots, 101\}$ such that no two elements differ by exactly 10 can be expressed as $p^a q^b r^c$ for some distinct primes p, q, r and positive integers a, b, c. Compute $(p + q + r)(a + b + c)$.

Proposed by Aaryan Vaishya

Answer: 13090

Equivalently, we want to choose a subset not containing any consecutive elements in each of the lists

$$\begin{gathered} 1, 11, \ldots, 91, 101 \\ 2, 12, \ldots, 98 \\ 3, 13, \ldots, 93 \\ \vdots \\ 10, 20, \ldots, 100 \end{gathered}$$

It is known that the number of ways to choose a subset of a set of n consecutive elements containing no consecutive elements is F_{n+2}; this can be proven by induction. The total number of ways is then

$$F_{13}F_{12}^9 = 233 \cdot 144^9 = 2^{36} \cdot 3^{18} \cdot 233^1$$

The desired value is then $(2 + 3 + 233)(36 + 18 + 1) = \boxed{13090}$.

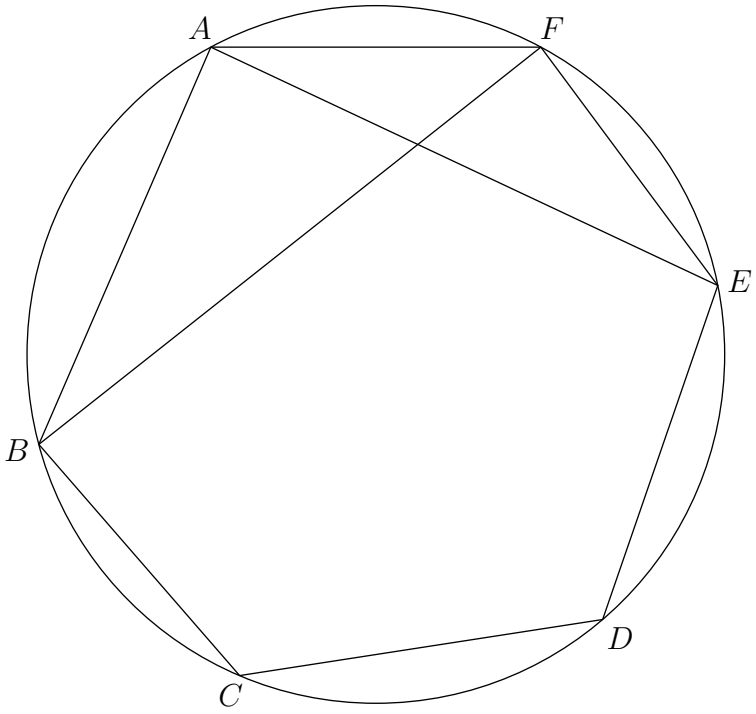
13. Cyclic hexagon $ABCDEF$, whose vertices lie in that order, is such that

$$\angle A, \angle B, \angle D, x, \angle C, \angle E, \angle F$$

forms an arithmetic sequence for some x. Also, $AE = 7$ and $BF = 8$. Compute the square of the circumradius of $ABCDEF$.

Proposed by Shrey Gupta and Darsh Patel

Answer: 19



Let the common difference of the arithmetic sequence be d. We have

$$\angle A + \angle C + \angle E = (180 - \frac{1}{2}\widehat{FAB}) + (180 - \frac{1}{2}\widehat{BCD}) + (180 - \frac{1}{2}\widehat{DEF}) = 540 - \frac{1}{2}(360) = 360$$

However, we also have,

$$\angle A + \angle C + \angle E = (x - 3\theta) + (x + \theta) + (x + 2\theta) = 3x$$

Then $x = 120$. Let R be the circumradius of $ABCDEF$. We have

$$\frac{7}{\sin(120 + 3\theta)} = \frac{AE}{\sin(\angle F)} = 2R = \frac{BF}{\sin(\angle A)} = \frac{8}{\sin(120 - 3\theta)}$$

Writing $\sin(120 + 3\theta) = \frac{\sqrt{3}}{2}\cos(3\theta) - \frac{1}{2}\sin(3\theta)$ and $\sin(120 - 3\theta) = \frac{\sqrt{3}}{2}\cos(3\theta) + \frac{1}{2}\sin(3\theta)$, this simplifies to

$$\sqrt{3}\cos(3\theta) = 15\sin(3\theta)$$

which yields

$$\cos(3\theta) = \frac{15}{\sqrt{228}}, \sin(3\theta) = \frac{\sqrt{3}}{\sqrt{228}}$$

Then

$$R = \frac{1}{2}\left(\frac{7}{\frac{\sqrt{3}}{2}\cos(3\theta) - \frac{1}{2}\sin(3\theta)}\right) = \sqrt{19}$$

and $R^2 = 19$.

14. Compute the number of positive integers $b \geq 2$ such that 81! has more trailing zeroes than 80! when both are written in base b.

Proposed by Aaryan Vaishya

Answer: 12720747185727

The number of trailing zeroes of n in base b is the largest exponent of a power of b that divides into n. Considering each prime $p \mid b$ individually, this value must be at most

$$\left\lfloor \frac{\nu_p(n)}{\nu_p(b)} \right\rfloor$$

and is the minimum of this value over all such p. Then for the given condition to be satisfied, we need the smallest values achieved by $p \mid b$ to increase from $\left\lfloor \frac{\nu_p(80!)}{\nu_p(b)} \right\rfloor$ to $\left\lfloor \frac{\nu_p(81!)}{\nu_p(b)} \right\rfloor$. A smallest value must then occur at $p \mid 81$, or $p = 3$. Then the unique minimum of $\left\lfloor \frac{\nu_p(80!)}{\nu_p(b)} \right\rfloor$ occurs at $p = 3$. We have

$$\nu_3(80!) = 36, \nu_3(81!) = 40$$

Then a multiple of $\nu_3(b)$ must occur between 37 and 40 inclusive. This yields the following possibilities for $\nu_3(b)$:

$$1, 2, 3, 4, 5, 8, 10, 13, 19, 20, 37, 38, 39, 40$$

Also, for all $p \mid b$ other than 3, we must have

$$\begin{aligned} &\left\lfloor \frac{\nu_p(80!)}{\nu_p(b)} \right\rfloor > \left\lfloor \frac{36}{\nu_3(b)} \right\rfloor \\ \iff &\frac{\nu_p(80!)}{\nu_p(b)} \geq \left\lfloor \frac{36}{\nu_3(b)} \right\rfloor + 1 \\ \iff &\nu_p(b) \leq \frac{\nu_p(80!)}{\left\lfloor \frac{36}{\nu_3(b)} \right\rfloor + 1} \\ \iff &\nu_p(b) \leq \left\lfloor \frac{\nu_p(80!)}{\left\lfloor \frac{36}{\nu_3(b)} \right\rfloor + 1} \right\rfloor \end{aligned}$$

This yields

$$\left\lfloor \frac{\nu_p(80!)}{\left\lfloor \frac{36}{\nu_3(b)} \right\rfloor + 1} \right\rfloor + 1$$

possibilities for any $p \neq 3$. Also, this is 1 for $p > 80$. We multiply the number of possibilities over all p for each possibility for $\nu_3(b)$, and sum:

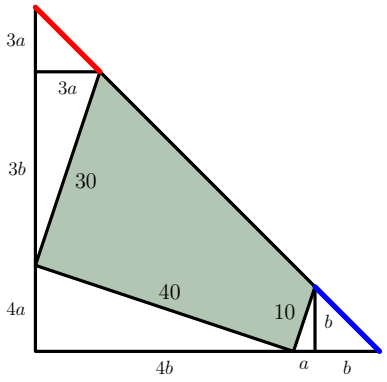
p		2	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	
$\nu_p(80!)$		78	19	12	7	6	4	4	3	2	2	2	1	1	1	1	1	1	1	1	1	1	
$\nu_3(b)$	$\left\lfloor \frac{36}{\nu_3(b)} \right\rfloor$										$\left\lfloor \frac{\nu_p(80!)}{\left\lfloor \frac{36}{\nu_3(b)} \right\rfloor + 1} \right\rfloor + 1$													Π
1	36	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3
2	18	5	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	10
3	12	7	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
4	9	8	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	32
5	7	10	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	60
8	4	16	4	3	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	768
10	3	20	5	4	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6400
13	2	27	7	5	3	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	68040
19	1	40	10	7	4	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	6451200
20	1	40	10	7	4	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	6451200
37	0	79	20	13	8	7	5	5	4	3	3	3	2	2	2	2	2	2	2	2	2	2	3180183552000
38	0	79	20	13	8	7	5	5	4	3	3	3	2	2	2	2	2	2	2	2	2	2	3180183552000
39	0	79	20	13	8	7	5	5	4	3	3	3	2	2	2	2	2	2	2	2	2	2	3180183552000
40	0	79	20	13	8	7	5	5	4	3	3	3	2	2	2	2	2	2	2	2	2	2	3180183552000
												Σ											12720747185727

15. A right trapezoid $ABCD$ with bases AB and CD and side BC perpendicular to its bases is inscribed in an isosceles right triangle so that A and D lie on the hypotenuse and B and C lie on different legs. Given that $AB = 30$, $BC = 40$, and $CD = 10$, compute the area of the isosceles right triangle.

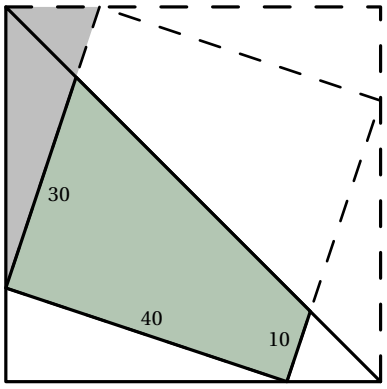
Proposed by Jacob Khohayting

Answer: 1280

Intuitively, we can simply rotate the isosceles right triangle to form a square inscribed in a square such that the diagonal of the larger square splits the side of the smaller square into lengths 10 and 30. To prove that this is "allowed", it suffices to prove that the highlighted segments on the diagonal are of equal length. This can be done by drawing the following perpendiculars:



Summing the distances on each side and equating them yields $b = 3a$. Therefore, the blue and red segments are equal lengths. Now, we obtain the following figure:



Notice, that the diagonal of the larger square treats as the angle bisector of the shaded right triangle. Since the diagonal splits the square into lengths 10 and 30, then by Angle Bisector Theorem, the legs of the right triangle are k and $3k$. Therefore, by Pythagorean Theorem, we have that $k^2 + (3k)^2 = 40^2$, so $k^2 = 160$. The isosceles right triangle has side lengths $4k$, so we want to find $8k^2$ which is $\boxed{1280}$.

16. Compute the sum of the squares of the side lengths of a non-degenerate, non-self-intersecting, convex 40-gon with smallest possible perimeter, all of whose vertices are lattice points.

Proposed by Aaryan Vaishya

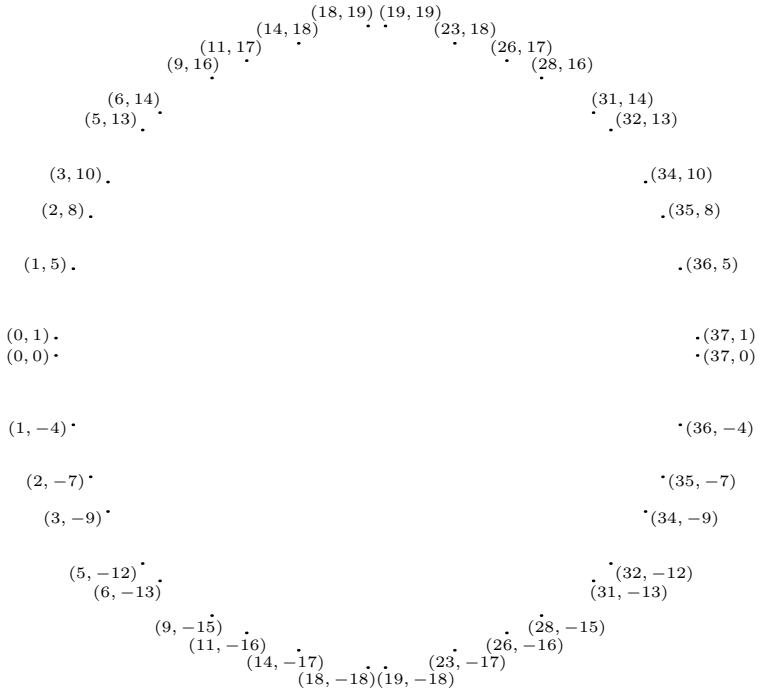
Answer: 372

Note that no 3 sides can be parallel. If a side has slope $\frac{a}{b}$, where $\frac{a}{b}$ is a reduced fraction, then the least possible length of that side is $\sqrt{a^2 + b^2}$. We will find 20 such distinct fractions with smallest possible value of $a^2 + b^2$, which, if there were a 40-gon using each of those fractions as slopes twice, would yield the desired 40-gon

with least perimeter. These fractions are

$$\begin{array}{cccc} & \frac{0}{1}, & \frac{1}{0} & \\ & \frac{1}{1}, & -\frac{1}{1} & \\ \frac{1}{2}, & \frac{2}{1}, & -\frac{2}{1}, & -\frac{1}{2} \\ \frac{1}{3}, & \frac{3}{1}, & -\frac{3}{1}, & -\frac{1}{3} \\ \frac{2}{3}, & \frac{3}{2}, & -\frac{3}{2}, & -\frac{2}{3} \\ \frac{1}{4}, & \frac{4}{1}, & -\frac{4}{1}, & -\frac{1}{4} \end{array}$$

We will provide a construction achieving the desired 40-gon. We can order the fractions in increasing order, with $\frac{1}{0}$ being last. Then we can use them for consecutive sides, with x-coordinate increasing, except for $\frac{1}{0}$, which has y-coordinate increasing. Then for the next 20 sides, we can use the negatives of the vectors that represent the first 20 sides, yielding the desired 40-gon.



The desired value is then

$$4(1) + 4(2) + 8(5) + 8(10) + 8(13) + 8(17) = \boxed{372}$$

17. Reals a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = 1$$
$$ab = cd$$

Compute the greatest possible value of

$$29a + 11b + 61c + 19d$$

Proposed by Darsh Patel

Answer: 70

The key observation is that such a, b, c, d can be parameterized by

$$\begin{aligned} a &= \cos(\alpha)\cos(\beta) \\ b &= \sin(\alpha)\sin(\beta) \\ c &= \cos(\alpha)\sin(\beta) \\ d &= \sin(\alpha)\cos(\beta) \end{aligned}$$

We can take α such that

$$\begin{aligned} a^2 + c^2 &= \cos^2(\alpha) \\ b^2 + d^2 &= \sin^2(\alpha) \end{aligned}$$

Then we can take β such that

$$\begin{aligned} a &= \cos(\alpha)\cos(\beta) \\ c &= \cos(\alpha)\sin(\beta) \end{aligned}$$

and from $ab = cd$ it follows that β is also such that

$$\begin{aligned} b &= \sin(\alpha)\sin(\beta) \\ d &= \sin(\alpha)\cos(\beta) \end{aligned}$$

and so this parametrization works. From product-to-sum, we have

$$\begin{aligned} a &= \frac{\cos(\theta) + \cos(\phi)}{2} \\ b &= \frac{-\cos(\theta) + \cos(\phi)}{2} \\ c &= \frac{\sin(\theta) - \sin(\phi)}{2} \\ d &= \frac{\sin(\theta) + \sin(\phi)}{2} \end{aligned}$$

where

$$\begin{aligned} \theta &= \alpha + \beta \\ \phi &= \alpha - \beta \end{aligned}$$

Then

$$29a + 11b + 61c + 19d = 9\cos(\theta) + 40\sin(\theta) + 20\cos(\phi) - 21\sin(\phi)$$

which has maximum value

$$\sqrt{9^2 + 40^2} + \sqrt{20^2 + (-21)^2} = \boxed{70}$$

18. Consider a positive integer n. In a game of musical chairs, there are n people sitting in chairs $1, 2, \ldots, n$. A permutation p of the chairs is chosen at random. Each player repeatedly travels from their current chair i to chair $p(i)$, taking 2^{-i} seconds to do so, until they end up at their original chair. Let E_n be the expected amount of time that elapses before everyone is done. The least real number r such that $E_n \leq r$ for all $n \geq 2$ can be expressed as $\frac{a}{b}$ for some relative prime positive integers a and b. Compute $100a + b$.

(Note that multiple people can be in a seat at the same time. Also, if $p(i) = i$, then the player starting in seat i is done after 2^{-i} seconds.)

Proposed by Aaryan Vaishya

Answer: 304

Consider any $n \geq 2$. The amount of time that elapses is the longest time it takes for any player to finish. However, we note that this is always achieved by the player starting at chair 1, and so we will find the expected number of time it takes for player 1 to finish.

The main claim is that each $i \neq 1$ is in the same cycle as 1 with probability $\frac{1}{2}$. We will count the permutations in which they are in the same cycle. Suppose there are k other elements of the cycle. We can choose them in $\binom{n-2}{k}$ ways. We can order the cycle in $(k+1)!$ ways and permute the rest of the chairs in $(n-k-2)!$ ways. Then the number of ways here is

$$\binom{n-2}{k}(k+1)!(n-k-2)! = (n-2)!(k+1)$$

Thus, the total number of ways is

$$\sum_{k=0}^{n-2}(n-2)!(k+1) = (n-2)!\left(\frac{n(n-1)}{2}\right) = \frac{n!}{2}$$

as desired.

The contribution from chair 1 is 2^{-1}, the expected contribution from the rest of the chairs is

$$\frac{1}{2}(2^{-2}+2^{-3}+\cdots+2^{-n})$$

by Linearity of Expectation. Then

$$E_n = 2^{-1}+\frac{1}{2}(2^{-2}+2^{-3}+\cdots+2^{-n}) = \frac{3}{4}-2^{-n-1}$$

This yields that $r = \frac{3}{4}$. The desired value is then $\boxed{304}$.

19. Let $O = (0,0)$ be the origin. Lattice points A, B, and C are such that their squared distances to the origin

$$OA^2, OB^2, OC^2$$

are consecutive integers in that order and are all less than 250. Compute the sum of all distinct possible values of OB^2.

Proposed by Darsh Patel

Answer: 559

Let

$$A = (x_1, y_1)$$
$$B = (x_2, y_2)$$
$$C = (x_3, y_3)$$

Let $a = x_2^2 + y_2^2$. Note that a sum of two squares x^2+y^2 can be $0, 1, 2, 4, 5 \pmod 8$. It follows that

$$a \equiv 1 \pmod 8$$

Also, x^2+y^2 can be $0, 1, 2, 4, 5, 7, 8 \pmod 9$. It follows that

$$a \equiv -1, 0, 1 \pmod 9$$

Then

$$a \equiv 1, 9, 17 \pmod{72}$$

We can now check all possible a, using Fermat Christmas Theorem to check whether $a-1, a, a+1$ are sums of two squares:

a	$a-1$?	a?	$a+1$?
1	✓	✓	✓
9	✓	✓	✓
17	✓	✓	✓
73	✓	✓	✓
81	✓	✓	✓
89	✗	✓	✓
145	✓	✓	✓
153	✗	✓	✗
161	✓	✗	✓
217	✗	✗	✓
225	✗	✓	✓
233	✓	✓	✓

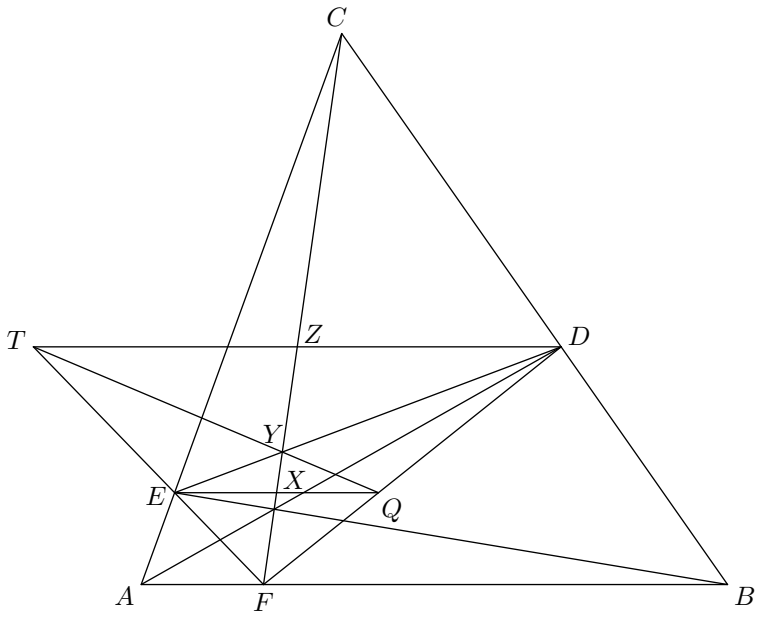
Then the sum of working a is

$$1+9+17+73+81+145+233=\boxed{559}$$

20. Consider triangle ABC and 30 points D, E, and F on sides BC, CA, and AB respectively such that AD, BE, and CF concur. Let lines k and l parallel to AB through E and D respectively intersect CF at X and Z respectively. Let line k intersect FD at Q and let lines CF and ED intersect at Y. If $\frac{FQ}{DQ}=\frac{17}{27}$, then $\frac{XF}{XY}$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a+b$.

Proposed by Xuzhou Ren

Answer: 6127



We use directed lengths. Since AD, BE, CF concur, we have

$$(B,C;D,BC\cap EF)=-1$$

Let $T=EF\cap DZ$ and ∞ be the point at infinity along AB. Then

$$(D,T;Z,\infty)\overset{F}{=}(D,BC\cap EF;C,B)=-1$$

It follows that Z is the midpoint of DT. Also,

$$(F,Y;X,Z)\overset{E}{=}(T,D;\infty,Z)=-1$$

It follows that

$$\frac{XF}{XY} = -\frac{ZF}{ZY}$$

Now, from

$$(D, T; Z, DT \cap EQ) = (D, T; Z, \infty) = -1$$

it follows that TQ passes through Y. Then by Menalaus Theorem on DFZ, we have

$$\frac{FY}{YZ} = -\frac{FQ}{QD} \cdot \frac{DT}{TZ} = -\frac{17}{27} \cdot (-2) = \frac{34}{27}$$

and

$$\frac{XF}{XY} = -\frac{ZF}{ZY} = -\left(1 + \frac{YF}{ZY}\right) = -\frac{61}{27}$$

Then $\frac{|XF|}{|XY|} = \frac{61}{27}$ and the desired value is $\boxed{6127}$.

21. A non-degenerate triangle is called *durable* if it has perimeter 1 and there exists a non-degenerate triangle whose side lengths are the lengths of its altitudes. The length of the smallest interval that contains each side length of every durable triangle can be expressed as $\frac{\sqrt{a}-b}{c}$ for some positive integers a, b, and c with a squarefree and $\gcd(b, c) = 1$. Compute $10000a + 100b + c$.

Proposed by Aaryan Vaishya

Answer: 50104

The upper bound is easily found by letting the triangle be isosceles, as the base approaches $\frac{1}{2}$, the congruent altitudes go to infinity and the altitude to the base goes to 0. We can obviously not go higher than $\frac{1}{2}$ for triangle inequality reasons.

For the lower bound, let a be the smallest side of the triangle, then we want b, c such that $\frac{1}{b} + \frac{1}{c} > \frac{1}{a}$ (altitudes are inversely proportional to their respective side lengths), $b, c < \frac{1}{2}$, $a + b + c = 1$. We can observe $\frac{1}{b} + \frac{1}{(1-a)-b}$ is increasing for $b > \frac{x}{2}$, so subject to the constraints it can obtain values only less than $\frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{2}-a}$. Thus we desire the value under 1, a such that $\frac{1}{a} = 2 + \frac{2}{1-2a}$, which resolves to $1 - 2a = 2(a)(1 - 2a) + 2a$, rearranging gives $4a^2 - 6a + 1$, whose relevant solution $\frac{6-\sqrt{36-16}}{8} = \frac{3-\sqrt{5}}{4}$.

The desired interval is then $\left(\frac{3-\sqrt{5}}{4}, \frac{1}{2}\right)$, giving the length as $\frac{\sqrt{5}-1}{4}$, so the requested sum is $\boxed{50104}$

22. A bijective function $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ is called *really-even* if there exists a positive integer k such that $f^{2^k}(x) = x$ is satisfied for all $x \in \{1, 2, \ldots, 8\}$. Compute the number of really-even f.

Proposed by Roger He

Answer: 11264

All f are just permutations of the integers 1 to 8. Separating permutation f into cycles, we see that f^k is the identity when the lcm of the cycle lengths divides k. Thus, the cycles of f are powers of 2, i.e. 1, 2, 4, and 8. Define $f(n)$ to be the number of really even permutations of the integers 1 to n, and $f(0) = 1$. The cycle of f that contains 1 has length 1, 2, 4, or 8. Thus, we have

$$f(8) = \binom{7}{0}(0!)f(7) + \binom{7}{1}(1!)f(6) + \binom{7}{3}(3!)f(4) + \binom{7}{7}(7!)f(0).$$

Similar formulas for $1, 2, \ldots, 7$ yield

$$\begin{aligned}
f(1) &= \binom{0}{0}(0!)f(0) \\
f(2) &= \binom{1}{0}(0!)f(1) + \binom{1}{1}(1!)f(0) \\
f(3) &= \binom{2}{0}(0!)f(2) + \binom{2}{1}(1!)f(1) \\
f(4) &= \binom{3}{0}(0!)f(3) + \binom{3}{1}(1!)f(2) + \binom{3}{3}(3!)f(0) \\
f(5) &= \binom{4}{0}(0!)f(4) + \binom{4}{1}(1!)f(3) + \binom{4}{3}(3!)f(1) \\
f(6) &= \binom{5}{0}(0!)f(5) + \binom{5}{1}(1!)f(4) + \binom{5}{3}(3!)f(2) \\
f(7) &= \binom{6}{0}(0!)f(6) + \binom{6}{1}(1!)f(5) + \binom{6}{3}(3!)f(3) \\
f(8) &= \binom{7}{0}(0!)f(7) + \binom{7}{1}(1!)f(6) + \binom{7}{3}(3!)f(4) + \binom{7}{7}(7!)f(0)
\end{aligned}$$

Then we can compute

$$\begin{aligned}
f(0) &= 1 \\
f(1) &= 1 \\
f(2) &= 2 \\
f(3) &= 4 \\
f(4) &= 16 \\
f(5) &= 56 \\
f(6) &= 256 \\
f(7) &= 1072 \\
f(8) &= \boxed{11264}
\end{aligned}$$

23. Complex numbers x and y satisfy

$$\begin{aligned}
x^3 + y^2 &= 8(x+1) \\
y^3 + x^2 &= 8(y+1)
\end{aligned}$$

Compute the sum of x^2 over all distinct possible values of x.

Proposed by Darsh Patel

Answer: 32

Let

$$\begin{aligned}
a &= x + y \\
b &= xy
\end{aligned}$$

Then x and y are the roots of $z^2 - az + b$. We compute

$$\begin{aligned}
&x^3 + y^2 - 8(x+1) \\
=&x(ax - b) + (ay - b) - 8(x+1) \\
=&a(ax - b) - bx + a(a - x) - b - 8(x+1) \\
=&(a^2 - a - b - 8)x + (a^2 - ab - b - 8)
\end{aligned}$$

Doing a similar computation for the second equation, we get

$$(a^2 - a - b - 8)x + (a^2 - ab - b - 8) = 0$$
$$(a^2 - a - b - 8)y + (a^2 - ab - b - 8) = 0$$

If $a^2 - a - b - 8 \neq 0$, we must have $x = y$. Then x is a root of $z^3 + z^2 - 8z - 8$. If $a^2 - a - b - 8 = 0$, we get

$$a^2 - a - b - 8 = 0$$
$$a^2 - ab - b - 8 = 0$$

Then $ab = a$ and either $a = 0$ or $b = 1$. If $a = 0$ then $b = -8$. If $b = 1$ then $a^2 - a - 9 = 0$. Denote the possible a in this case by a_1, a_2. Then the possible x are the roots of any of

$$z^3 + z^2 - 8z - 8, z^2 - 8, z^2 - a_1 z + 1, z^2 - a_2 z + 1$$

Now, $z^2 - 8$ is a factor of $z^3 + z^2 - 8z - 8$ and so we can ignore it:

$$z^3 + z^2 - 8z - 8, z^2 - a_1 z + 1, z^2 - a_2 z + 1$$

We can check that none of these polynomials have double roots. We can also check that no pair of these polynomials have a non-constant common divisor. Then the sum of x^2 over all possible x is

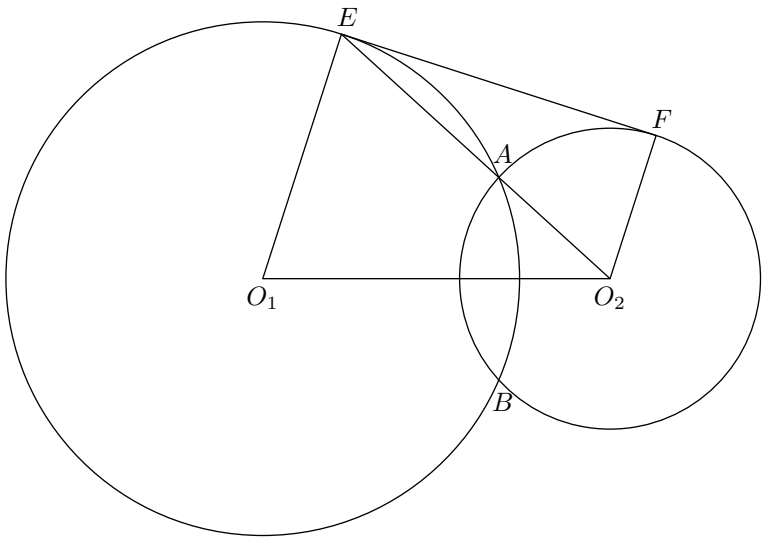
$$((-1)^2 - 2(-8)) + (a_1^2 - 2) + (a_2^2 - 2)$$

Computing $a_1^2 + a_2^2 = 19$, we get that this is equal to $\boxed{32}$.

24. Circles ω_1, ω_2 centered at O_1, O_2 respectively intersect at distinct points A and B. An external tangent of the circles meets ω_1 and ω_2 at E and F respectively such that point A is closer to segment EF. Given that E, A, O_2 are collinear and $EA = 3$, the minimum possible sum of the radii of two circles can be expressed as $\frac{a\sqrt{b}+c}{d}$ for some positive integers a, b, c, and d with b squarefree and $\gcd(a, c, d) = 1$. Compute $1000000a + 10000b + 100c + d$.

Proposed by Tom Zhang

Answer: 6020302



Denote the radii of two circles as r_1, r_2 respectively. Denote the length of EF as l, the length of O_1O_2 as d. By power of point, we have $r_2(r_2 + 3) = d^2 - r_1^2$, by external tangent formula, $d^2 - (r_2 - r_1)^2 = l^2$. Moreover, by Pythagorean theorem in $\triangle EFO_2$, $l^2 + r_2^2 = (3 + r_2)^2$.

Generalize these equations, we have
$9+6r_2 = l^2; r_1^2+r_2^2-2r_1r_2+l^2 = d^2 = r_1^2+r_2^2+3r_2$.
Thus, we have $l^2 = 3r_2+2r_1r_2 = 9+6r_2, 3r_2+9 = 2r_1r_2 \implies r_2(2r_1-3) = 9, r_1+r_2 = \frac{9}{2r_1-3}+r_1 = \frac{9}{2r_1-3}+r_1-\frac{3}{2}+\frac{3}{2} \geq 2\sqrt{\frac{9}{2}}+\frac{3}{2} = 3\sqrt{2}+\frac{3}{2} = \frac{6\sqrt{2}+3}{2} \implies \boxed{6020302}$.

25. Compute the area of the finite region bounded by the graph of

$$y = |||\ldots||||x-0|-1|-2|-3|\ldots|-199|-200|$$

and the x-axis.

Proposed by Darsh Patel

Answer: 4030000

We will consider the area bounded by the graph of

$$y = |||\ldots|||x-1|-2|-3|\ldots|-199|-200|$$

and the x-axis over $x \geq 0$. The desired area is twice this amount.

We first note that this function locally behaves like $x+c$ or $-x+c$ for some constant c, which is based on whether each absolute value in the expression is equal to its argument or the negative of its argument. In particular, there exist $s_1, s_2, \ldots, s_k, v_1, v_2, \ldots, v_k, v_{k+1}$ such that

$$y = \begin{cases} (-1)^k x + v_1 & 0 \leq x \leq s_1 \\ (-1)^{k-1}x + v_2 & s_1 \leq x \leq s_2 \\ (-1)^{k-2}x + v_3 & s_2 \leq x \leq s_3 \\ \vdots & \vdots \\ -x+v_k & s_{k-1} \leq x \leq s_k \\ x+v_{k+1} & s_k \leq x \end{cases}$$

We will find $s_1, s_2, \ldots, s_k$.

We note that at each s_i the argument of some absolute value changes sign. This can only occur if the argument of that absolute value is 0 at s_i. We see

$$\begin{aligned} &||||\ldots|||x-1|-2|-3|\ldots|-(j-3)|-(j-2)|-(j-1)|-j = 0 \\ \implies &|||\ldots|||x-1|-2|-3|\ldots|-(j-3)|-(j-2)| = j+(j-1) \\ \implies &||\ldots|||x-1|-2|-3|\ldots|-(j-3)| = j+(j-1)+(j-2) \\ &\vdots \\ \implies &x = j+(j-1)+(j-2)+\cdots+1 \end{aligned}$$

Then each s_i is some triangular number $T_j = 1+2+\cdots+j$. In particular, it follows that between consecutive triangular numbers, the graph y is a line segment. We can then find the area under y over this interval as the area of a trapezoid.

Consider when x is a triangle number T_j. Then

$$\begin{aligned}
&|||\dots|||x-1|-2|-3|\dots|-199|-200| \\
=&|||\dots||||0-(j+1)|-(j+2)|-(j+3)|-(j+4)|\dots|-199|-200| \\
=&|||\dots|||(j+1)-(j+2)|-(j+3)|-(j+4)|\dots|-199|-200| \\
=&|||\dots||1-(j+3)|-(j+4)|\dots|-199|-200| \\
=&|||\dots|(j+2)-(j+4)|\dots|-199|-200| \\
=&\begin{cases} \frac{200-j}{2} & 2 \nmid 201-j \\ j+\frac{201-j}{2} & 2 \mid 201-j \end{cases} \\
=&\begin{cases} \frac{200-j}{2} & 2 \mid j \\ \frac{201+j}{2} & 2 \nmid j \end{cases}
\end{aligned}$$

We also note that $y = x - T_{200}$ for $x \geq T_{200}$, and so the desired area is the area under y from $x = 0$ to $x = T_{200}$. We now consider the area bounded under y between T_j and T_{j+1} for $0 \leq j \leq 199$. This is

$$\frac{(T_{j+1}-T_j)\left(\frac{200-j}{2}+\frac{202+j}{2}\right)}{2} = \frac{201(j+1)}{2}$$

if j is even and

$$\frac{(T_{j+1}-T_j)\left(\frac{201+j}{2}+\frac{199-j}{2}\right)}{2} = \frac{200(j+1)}{2}$$

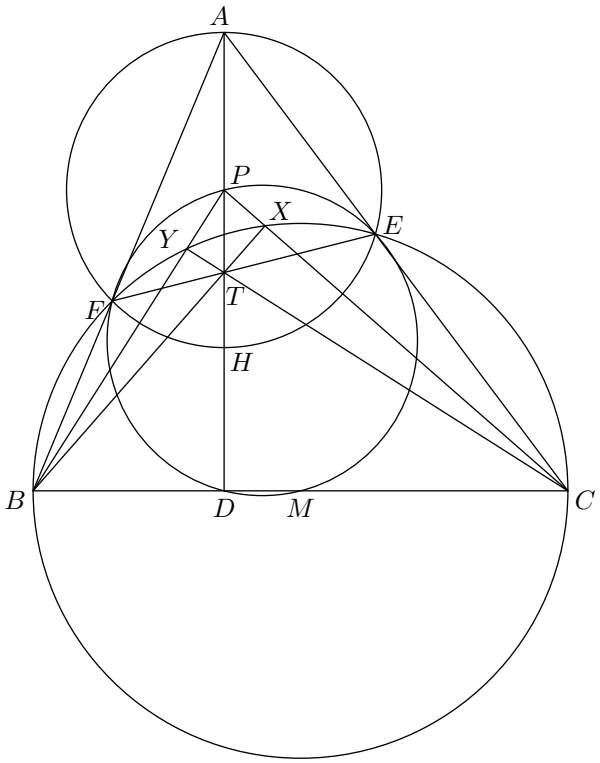
if j is odd. The desired area is then

$$\begin{aligned}
&2\left(\left(\sum_{\substack{2|j \\ 0\leq j\leq 199}} \frac{201(j+1)}{2}\right)+\left(\sum_{\substack{2\nmid j \\ 0\leq j\leq 199}} \frac{200(j+1)}{2}\right)\right) \\
=&\left(\sum_{\substack{2|j \\ 0\leq j\leq 199}} 201(j+1)\right)+\left(\sum_{\substack{2\nmid j \\ 0\leq j\leq 199}} 200(j+1)\right) \\
=&\left(\sum_{\substack{2|j \\ 0\leq j\leq 199}} (j+1)\right)+200\left(\sum_{0\leq j\leq 199} (j+1)\right) \\
=&100^2+200\left(\frac{(200)(201)}{2}\right) \\
=&\boxed{4030000}
\end{aligned}$$

26. Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the feet of the A, B, and C-altitudes respectively. Let T be the intersection of lines AD and EF. Let P be the orthocenter of BTC. The length PT can be expressed as $\frac{a}{b}$, for some relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Darsh Patel

Answer: 12156



Let H be the orthocenter of ABC and M be the midpoint of BC. Let X and Y be the feet of the perpendiculars from B and C to CT and BT respectively. Then P is the intersection of lines BY and CX. We will show that P is the midpoint of AH. Let P' be this midpoint. The main claim is that under inversion with respect to the circumcircle of AEF, which we denote as ω, $Y = B^*$ and $X = C^*$. Note that P' is the circumcenter of AEF.

We see that $BCEF$ is cyclic with circumcenter M. Let Γ be its circumcircle. We claim that ω and Γ are orthogonal. We have that

$$\begin{aligned}
&\angle MEP' \\
=&\angle MEF + \angle FEP' \\
=&\left(90 - \frac{1}{2}\angle EMF\right) + \left(90 - \frac{1}{2}\angle EP'F\right) \\
=&180 - \angle EBF - \angle EAF \\
=&180 - (90 - \angle BAC) - \angle BAC \\
=&90
\end{aligned}$$

as desired. Then the image of Γ under the inversion is itself, and so B^* lies on Γ. We also note that P', E, F, and D all lie on the 9-point circle ω_9 of ABC. Now, the image of ω_9 is line EF, and so $D^* = T$. It now follows that $BDTB^*$ is cyclic. Then

$$\angle BB^*T = 180 - \angle BDT = 90$$

and so B^*, T, and C are collinear. Then B^* is the second intersection of CT with Γ. However, this is exactly Y, and so $Y = B^*$ and P' lies on BY. Similarly, P' also lies on CX, and so $P' = P$.

Let R be the circumradius of ABC. Since $BCFE$ is cyclic, we have

$$AFE \sim ABC$$

and so

$$|PA| = \frac{|AE|}{|AB|} \cdot R = \frac{\frac{33}{5}}{13} \cdot \frac{65}{8} = \frac{33}{8}$$

Then

$$|PT| = \frac{|PA|^2}{|PD|} = \frac{|PA|^2}{|AD| - |PA|} = \frac{\left(\frac{33}{8}\right)^2}{12 - \frac{33}{8}} = \frac{121}{56}$$

The desired value is then $\boxed{12156}$.

27. Permutations x and y of the word $ABCDE$ are chosen randomly. Let $F(x, y)$ be the length of the shortest string that contains both permutations as substrings. The expected value of $F(x, y)$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Roger He

Answer: 113420

Replace the letters A, B, C, D, E with the numbers $1, 2, 3, 4, 5$ respectively. Let $x = 12345$. Any string y can either go on the end of x, before x, or it could do both. For example, 45231 could merge with x as 12345231 or 452312345. Let $S(m, n)$ be the set of strings that can share m letters at the beginning of x, or n letters at the end. 45231 would belong in $S(1, 2)$. When $m = n = 5$, then $|S(5, 5)| = 1$, since it is just 12345. When $0 < m$, $n \leq 4$, then $|S(m, n)| = (5 - m - n)!$. When $m = 0$ or $n = 0$, then

$$|S(0, n)| = (5 - n)! - (5 - n - 1)! - \ldots - 1! - 0!.$$

To get $|S(0, 0)|$, subtract the rest of the $|S|$ from 120. The length of the shortest string containing x and y for any string $y \in S(m, n)$ is $10 - \max(m, n)$. Summing $(10 - \max(m, n))|S(m, n)|$ over all pairs m, n, we find the total sum of string lengths over all y to be 1133. Dividing by 120, we get the expected value over (x, y) where $x = 12345$. Notice that all strings x are functionally identical to 12345, thus the expected value over all pairs (x, y) is the same as the expected value over $(12345, y)$. Thus, the expected value is $\frac{1133}{120}$. The desired value is then $\boxed{113420}$.

28. Let $\omega = \text{cis}\left(\frac{2\pi}{7}\right)$. A polynomial $P(x)$ with rational coefficients has minimum possible degree while satisfying

$$P(\omega) = \frac{1}{(2 - \omega)^2}$$

Then $P(1)$ can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Darsh Patel

Answer: 1315429

In order to turn $\frac{1}{(2-\omega)^2}$ into a polynomial in ω, we express it as a sum:

$$\begin{aligned}
&\frac{1}{(2 - \omega)^2} \\
=&\frac{1}{2^2}\left(\frac{1}{\left(1 - \frac{\omega}{2}\right)^2}\right) \\
=&\frac{1}{2^2} + \frac{2\omega}{2^3} + \frac{3\omega^2}{2^4} + \cdots \\
=&\left(\frac{1}{2^2} + \frac{8}{2^9} + \cdots\right) + \left(\frac{2}{2^3} + \frac{9}{2^{10}} + \cdots\right)\omega + \cdots + \left(\frac{7}{2^8} + \frac{14}{2^{15}} + \cdots\right)\omega^6
\end{aligned}$$

Define

$$\begin{aligned} a_0 &= \frac{1}{2^2} + \frac{8}{2^9} + \cdots \\ a_1 &= \frac{2}{2^3} + \frac{9}{2^{10}} + \cdots \\ &\vdots \\ a_6 &= \frac{7}{2^8} + \frac{14}{2^{15}} + \cdots \end{aligned}$$

We have

$$a_0 + a_1\omega + \cdots + a_6\omega^6 = (a_0 - a_6) + (a_1 - a_6)\omega + \cdots + (a_5 - a_6)\omega^5$$

Then we can take

$$P(x) = (a_0 - a_6)x + \cdots + (a_5 - a_6)x^5$$

We claim that this is exactly the desired $P(x)$. If there were another possibility $Q(x)$ of degree at most 5, we would have

$$P(\omega) - Q(\omega) = 0$$

However, the minimal polynomial of ω has degree 6, from which it follows that $P(x) = Q(x)$, as desired. Now, we have

$$a_0 + a_1 + \cdots + a_6 = \frac{1}{(2-1)^2} = 1$$

and

$$a_6 = \frac{7}{2^8}\left(1 + \frac{2}{2^7} + \cdots\right) = \frac{7}{2^8}\left(\frac{1}{\left(1 - \frac{1}{2^7}\right)^2}\right)$$

Then

$$P(1) = (a_0 + a_1 + \cdots + a_6) - 7a_6 = 1 - \frac{7^2}{2^8}\left(\frac{1}{\left(1 - \frac{1}{2^7}\right)^2}\right) = \frac{12993}{16129}$$

The desired value is then $\boxed{1315429}$.

29. For any positive integer n and real r, let $f(n, r)$ be the least positive integer k for which there exist relatively prime perfect squares a and b with

- $a + b = n^k$
- $|a - b| \leq r(n^k)$

or 0 if no such k exists. Given n such that $f\left(n, \frac{1}{2}\right) = 2024$, compute $f\left(n, \sqrt{\frac{1}{2}}\right)$.

Proposed by Darsh Patel

Answer: 1518

Note that if n is divisible by a prime p with $p = 2$ or $p \equiv 3 \pmod 4$, we cannot have $f\left(n, \frac{1}{2}\right) \geq 2$. So, n only has prime divisors p with $p \equiv 1 \pmod 4$. Also, $n \neq 1$.

Consider $f(n, r) = k$ for any $r > 0$, and let a, b be as in the definition of f. Let

$$a = x^2, b = y^2$$

Since n has only prime divisors p with $p \equiv 1 \pmod 4$, all of its irreducible factors in $\mathbb{Z}[i]$ are of the form $s+ti$, where s^2+t^2 is a 1 modulo 4 prime. Then n can be factored in $\mathbb{Z}[i]$ as

$$(s_1+t_1i)^{d_1}(s_1-t_1i)^{d_1}(s_2+t_2i)^{d_2}(s_2-t_2i)^{d_2}\dots(s_u+t_ui)^{d_u}(s_u-t_ui)^{d_u}$$

for some positive integers $s_1, t_1, d_1, s_2, t_2, d_2, \dots, s_u, t_u, d_u$, where s_j+t_ji and s_j-t_ji are irreducible. Then since

$$(x+yi)(x-yi) = n^k$$

and $\mathbb{Z}[i]$ is a unique factorization domain and $\gcd(x,y)=1$ we have that

$$x+yi = \alpha(s_1 \pm t_1i)^{d_1k}(s_2 \pm t_2i)^{d_2k}\dots(s_u \pm t_ui)^{d_uk}$$

for some unit α and some choice of $\pm$s. Let θ_j be an argument of s_j+t_ji. Then in polar form,

$$x+yi = \sqrt{n^k}\operatorname{cis}\left(\frac{\ell\pi}{2} + k\left(\pm d_1\theta_1 \pm d_2\theta_2 \pm \dots \pm d_u\theta_u\right)\right)$$

for some integer ℓ. Then

$$\begin{aligned}
&|a-b| \\
=&|\Re((x+yi)^2)| \\
=&n^k\left|\cos\left(\ell\pi + 2k\left(\pm d_1\theta_1 \pm d_2\theta_2 \pm \dots \pm d_u\theta_u\right)\right)\right| \\
=&n^k\left|\cos(k(\pm 2d_1\theta_1 \pm 2d_2\theta_2 \pm \dots \pm 2d_u\theta_u))\right|
\end{aligned}$$

Then k is the least value for which

$$\left|\cos(k(\pm 2d_1\theta_1 \pm 2d_2\theta_2 \pm \dots \pm 2d_u\theta_u))\right| \le r$$

for some choice of $\pm$s. Let $\phi_1, \phi_2, \dots, \phi_{2^u}$ be the angles corresponding to the choices of $\pm$s in

$$\pm 2d_1\theta_1 \pm 2d_2\theta_2 \pm \dots \pm 2d_u\theta_u$$

and let $\psi_1, \psi_2, \dots, \psi_{2^u}$ be such that ψ_j is the unique angle in $\left[0, \frac{\pi}{2}\right]$ with $\cos(\psi_j) = |\cos(\phi_j)|$. Note that $|\cos(k\psi_j)| = |\cos(k\phi_j)|$ for all k. Then k is the least value for which

$$|\cos(k\psi_j)| \le r$$

for some j.

Then since $f\left(n, \frac{1}{2}\right) = 2024$, 2024 is the least k with $|\cos(k\psi_j)| \le \frac{1}{2}$ for some j. For fixed j, the least k with $|\cos(k\psi_j)| \le \frac{1}{2}$ is the least k with $k\psi_j \ge \frac{\pi}{3}$. Let

$$\psi_m = \max\{\psi_1, \psi_2, \dots, \psi_{2^u}\}$$

Then 2024 is the least k with $k\psi_m \ge \frac{\pi}{3}$ and we have

$$\frac{1}{2024}\left(\frac{\pi}{3}\right) \le \psi_m < \frac{1}{2023}\left(\frac{\pi}{3}\right)$$

Now, $f\left(n, \sqrt{\frac{1}{2}}\right)$ is the least k with $k\psi_m \ge \frac{\pi}{4}$. Since

$$\frac{1}{1518}\left(\frac{\pi}{4}\right) \le \psi_m < \frac{1}{1517+\frac{1}{4}}\left(\frac{\pi}{4}\right)$$

we have $f\left(n, \sqrt{\frac{1}{2}}\right) = \boxed{1518}$.

We remark that 611531820025012^{2024} is an example of n that works.

30. Consider cyclic hexagon $ABDEFC$ with $AB = AC$. Define $P = AD \cap BE$, $Q = CD \cap BF$, $R = AF \cap CE$, $S = AD \cap BF$, and $T = AF \cap CD$. Point K lies on line TS such that $\angle SKQ = \angle ACE$. Given that $AB = 20$, $PQ = 7$, $PR = 18$, and $SK = 5$, TK can be expressed as $\frac{a}{b}$ for some relatively prime positive integers a and b. Compute $100a + b$.

Proposed by Tom Zhang

Answer: 5507

As $AC = AB, \angle AFC = \angle AFB = \angle ADC$, which tells that S, T, F, D are concyclic. Notice that since F, C, B, D are also concyclic, which tells that $CB||TS$ by Reim. Then, $\angle FAC = \angle CBF = \angle TSQ; \angle SKQ = \angle ACE, \angle ARC = \angle SQK$, similarly, $\angle TQK = \angle EPD = \angle APB$

We then realize that $\frac{\sin\angle SQK}{\sin\angle TQK} = \frac{\sin\angle ARC}{\sin\angle APB} = \frac{\sin\angle ARC}{\sin\angle ACR} \cdot \frac{\sin\angle ABP}{\sin\angle BPA} = \frac{AC}{AR} \cdot \frac{AP}{AB} = \frac{AP}{AR}$(Note that since $ACEB$ is a cyclic quadrilateral, $\angle ACE + \angle ABE = 180°$) that is why we use $\sin\angle ACR, \sin\angle ABP$ to cancel each other)

By Pascal, we know P, Q, R are collinear, $\frac{\sin\angle ARP}{\sin\angle APR} = \frac{AP}{AR}$. Then by LOS again, we have $\frac{SQ}{PQ} = \frac{\sin\angle APR}{\sin\angle PSQ}; \frac{TQ}{RQ} = \frac{\sin\angle ARP}{\sin\angle RTQ}$, so we have $\frac{SQ}{TQ} = \frac{PQ}{RQ} \cdot \frac{\sin\angle SPQ}{\sin\angle PSQ} \cdot \frac{\sin\angle RTQ}{\sin\angle TRQ} = \frac{PQ}{RQ} \cdot \frac{\sin\angle SPQ}{\sin\angle TRQ} = \frac{PQ}{RQ} \cdot \frac{AP}{AR}$(since F, T, S, D are concyclic, $\angle PSQ = \angle RTQ$)

Now we also realize that $\frac{SK}{TK} = \frac{SQ}{TQ} \cdot \frac{\sin\angle SQK}{\sin\angle TQK} = \frac{SQ}{TQ} \cdot \frac{AP}{AR} = \frac{PQ}{RQ}$. Now we plug the numbers in, $TK = \frac{SK \cdot RQ}{PQ} = \frac{5 \cdot (18-7)}{7} = \frac{55}{7}$ leads to $\boxed{5507}$.