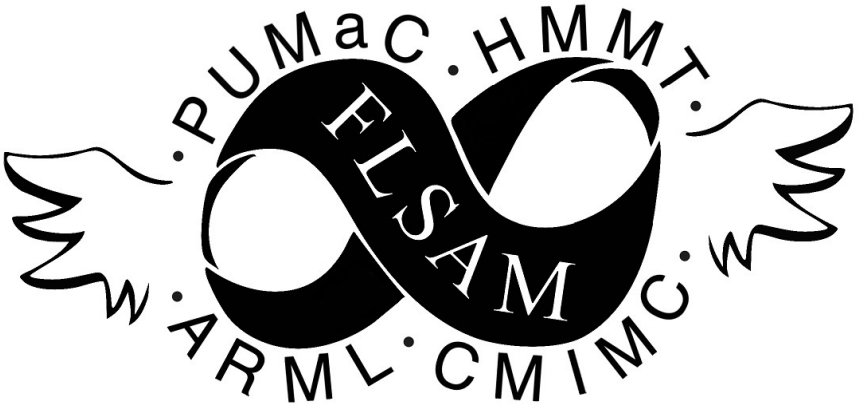
Florida Student Association of Mathematics



2025 HMMT Tryout

Winter 2024-2025

Solutions

Algebra and Number Theory Answers

1. $-\frac{11}{3}$
2. $\frac{29}{10}$
3. 20752
4. 946
5. 183
6. 24

Geometry Answers

1. 40
2. $\frac{71}{15}$
3. 18
4. $12\sqrt{2}+4\sqrt{7}, 12\sqrt{2}-4\sqrt{7}$
5. $\frac{595\sqrt{35}}{81}$
6. 9

Combinatorics Answers

1. $\frac{217}{12}$
2. $\frac{2}{105}$
3. 4
4. 10
5. $\frac{4\pi}{21}$
6. $\frac{1}{12}$

Algebra and Number Theory

1. A cubic polynomial $P(x)$ with rational coefficients is such that each of

$$P(\sqrt{2})+\sqrt{2}$$
$$\sqrt{3}P(\sqrt{3})+\sqrt{3}$$
$$P(\sqrt{5})-\sqrt{5}$$
$$\sqrt{7}P(\sqrt{7})-\sqrt{7}$$

are rational. Compute $P(1)$.

Proposed by Aaryan Vaishya & Darsh Patel

Answer: $-\frac{11}{3}$

Write

$$P(x)=ax^3+bx^2+cx+d$$

Then

$$P(\sqrt{2})+\sqrt{2}=(2b+d)+(2a+c+1)\sqrt{2}$$
$$\sqrt{3}P(\sqrt{3})+\sqrt{3}=3(3a+c)+(3b+d+1)\sqrt{3}$$
$$P(\sqrt{5})-\sqrt{5}=(5b+d)+(5a+c-1)\sqrt{5}$$
$$\sqrt{7}P(\sqrt{7})-\sqrt{7}=7(7a+c)+(7b+d-1)\sqrt{7}$$

and

$$2a+c+1=0$$
$$3b+d+1=0$$
$$5a+c-1=0$$
$$7b+d-1=0$$

Solving, we get

$$(a,b,c,d)=\left(\frac{2}{3},\frac{1}{2},-\frac{7}{3},-\frac{5}{2}\right)$$

Then

$$P(1)=a+b+c+d=-\frac{11}{3}$$

2. Positive reals x and y satisfy

$$\sqrt{x}+\sqrt{y}=2$$
$$\sqrt{x+1}+\sqrt{y+1}=3$$

Compute $x+y$.

Proposed by Tom Zhang

Answer: $\frac{29}{10}$

Squaring both sides of both equations, we get

$$x+y+2\sqrt{xy}=4$$
$$x+y+2\sqrt{xy+x+y+1}=7$$

Let $a = x + y$ and $b = xy$. Then

$$a + 2\sqrt{b} = 4$$
$$a + 2\sqrt{a + b + 1} = 7$$

Substituting $a = 4 - 2\sqrt{b}$ into the second equation, we get

$$\begin{aligned} & a + 2\sqrt{a + b + 1} = 7 \\ \implies & (4 - 2\sqrt{b}) + 2\sqrt{b - 2\sqrt{b} + 5} = 7 \\ \implies & 2\sqrt{b - 2\sqrt{b} + 5} = 2\sqrt{b} + 3 \\ \implies & 4(b - 2\sqrt{b} + 5) = 4b + 12\sqrt{b} + 9 \\ \implies & \sqrt{b} = \frac{11}{20} \end{aligned}$$

Then

$$x + y = a = 4 - 2\sqrt{b} = \frac{29}{10}$$

3. Consider the polynomial $P(x) = x^{400} + 12x + 1$, having roots $r_1, r_2 \ldots r_{400}$. Let $Q(x)$ be the monic polynomial whose roots are exactly $r_i^4 + 2$ for $i = 1, 2, \ldots, 400$. Compute $Q(1)$.

Proposed by Aaryan Vaishya

Answer: 20752

We have

$$Q(1) = (1 - (r_1^4 + 2)) \ldots (1 - (r_{400}^4 + 2)) = (r_1^4 + 1) \ldots (r_{400}^4 + 1)$$

Let a, b, c, d be the roots of $x^4 + 1$. Then this equals

$$(r_1 - a)(r_1 - b)(r_1 - c)(r_1 - d) \ldots (r_{400} - a)(r_{400} - b)(r_{400} - c)(r_{400} - d) = P(a)P(b)P(c)P(d)$$

Now, we have

$$P(a) = a^{400} + 12a + 1 = 12a + 2$$

and similar results for b, c, d. Then this equals

$$(12a + 2) \ldots (12d + 2) = 12^4 \left(-\frac{1}{6} - a\right) \ldots \left(-\frac{1}{6} - d\right) = 12^4 \left(\left(-\frac{1}{6}\right)^4 + 1\right) = 20752$$

4. Let $f(x) = x^2$. For an integer n, we let $d(n)$ be the least nonnegative integer k for which

$$\underbrace{f(f(\ldots f}_{k+1\ f\text{'s}}(n)\ldots)) \equiv \underbrace{f(f(\ldots f}_{k\ f\text{'s}}(n)\ldots)) \pmod{256}$$

Compute $\sum_{n=1}^{256} d(n)$.

Proposed by Darsh Patel

Answer: 946

We have

$$f^{k+1}(n) - f^k(n) = n^{2^k}(n^{2^k} - 1)$$

We will find $d(n)$ for each n.

Suppose $n \equiv 0 \pmod 2$. Then we need $\nu_2(n^{2^k}) \geq 8$, which is equivalent to

$$2^k \nu_2(n) \geq 8$$

Then $k = 0, 1, 2, 3$ require

$$\nu_2(n) \in \{8\}, \{4, 5, 6, 7\}, \{2, 3\}, \{1\}$$

respectively and have

$$1, 15, 48, 64$$

cases respectively.

Suppose $n \equiv 1 \pmod 4$. Then we need $\nu_2(n^{2^k} - 1) \geq 8$, which is equivalent to

$$\nu_2(n-1) + k \geq 8$$

by the Lifting the Exponent Lemma. Then $k = 0, 1, 2, 3, 4, 5, 6$ require

$$\nu_2(k-1) = 8 - k$$

and have

$$1, 1, 2, 4, 8, 16, 32$$

cases respectively.

Suppose $n \equiv 3 \pmod 4$. Under the map $n \to -n$, this is symmetric to the case of $n \equiv 1 \pmod 4$, except for $n = 255$, which has $d(255) = 2$, while $d(1) = 1$.

Now we can compute

$$\sum_{n=1}^{256} d(n) = 0(1+2) + 1(15+2) + 2(48+4) + 3(64+8) + 4(16) + 5(32) + 6(64) + 1 = 946$$

5. Let a, b, c be the roots of the polynomial $x^3 - x^2 - 2x + 1$. Let S be the set of triples of nonnegative integers (i, j, k) with $i + j + k = 9$ and $ijk = 0$. Compute

$$\sum_{(i,j,k) \in S} a^i b^j c^k$$

Proposed by Darsh Patel

Answer: 183

We can instead think of S as the multiset

$$\begin{aligned}&\{(0,9,0), (0,8,1), \ldots, (0,0,9)\} + \\ &\{(0,0,9), (1,0,8), \ldots, (9,0,0)\} + \\ &\{(9,0,0), (8,1,0), \ldots, (0,9,0)\} - \\ &\{(9,0,0), (0,9,0), (0,0,9)\}\end{aligned}$$

Now define S_n the same way as S except with 9 replaced by n; then $S = S_9$. Also define

$$s_n = \sum_{(i,j,k) \in S_n} a^i b^j c^k$$

We have that

$$s_n = \left(\sum_{\text{cyc}}(a^n + a^{n-1}b + \cdots + b^n)\right) - \left(\sum_{\text{cyc}} a^n\right) = \left(\sum_{\text{cyc}} \frac{a^{n+1} - b^{n+1}}{a-b}\right) - \left(\sum_{\text{cyc}} a^n\right) = \sum_{\text{cyc}} \left(\frac{a}{a-b} + \frac{a}{a-c} - 1\right) a^n$$

It follows that $\{s_n\}$ is a linear recurrence with characteristic polynomial $x^3 - x^2 - 2x + 1$. Then we can find s_9 by finding s_0, s_1, s_2 and applying the recurrence. We have

$$s_0 = 0$$
$$s_1 = a + b + c = 1$$
$$s_2 = a^2 + ab + b^2 + bc + c^2 + ca = (a+b+c)^2 - (ab + bc + ca) = 3$$

Since

$$s_n = s_{n-1} + 2s_{n-2} - s_{n-3}$$

we can now compute

$$s_0 = 0$$
$$s_1 = 1$$
$$s_2 = 3$$
$$s_3 = 5$$
$$s_4 = 10$$
$$s_5 = 17$$
$$s_6 = 32$$
$$s_7 = 56$$
$$s_8 = 103$$
$$s_9 = 183$$

6. Positive integers a_0, a_1, a_2, a_3, a_4 are such that

$$a_0a_0 + a_1a_4 + a_2a_3 + a_3a_2 + a_4a_1 = 187$$
$$a_0a_1 + a_1a_0 + a_2a_4 + a_3a_3 + a_4a_2 = 175$$
$$a_0a_2 + a_1a_1 + a_2a_0 + a_3a_4 + a_4a_3 = 184$$
$$a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0 + a_4a_4 = 179$$
$$a_0a_4 + a_1a_3 + a_2a_2 + a_3a_1 + a_4a_0 = 175$$

Compute the sum of the possible values of $a_0 + a_1$.

Proposed by Darsh Patel

Answer: 24

The key idea is that, defining $\omega = \text{cis}\left(\frac{2\pi}{5}\right)$, we have

$$\begin{aligned}(a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4)^2 =& (a_0a_0 + a_1a_4 + a_2a_3 + a_3a_2 + a_4a_1) + \\ & (a_0a_1 + a_1a_0 + a_2a_4 + a_3a_3 + a_4a_2)\omega + \\ & (a_0a_2 + a_1a_1 + a_2a_0 + a_3a_4 + a_4a_3)\omega^2 + \\ & (a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0 + a_4a_4)\omega^3 + \\ & (a_0a_4 + a_1a_3 + a_2a_2 + a_3a_1 + a_4a_0)\omega^4\end{aligned}$$

Now, this equals

$$187 + 175\omega + 184\omega^2 + 179\omega^3 + 175\omega^4 = 12 + 9\omega^2 + 4\omega^3 = (3\omega + 2\omega^4)^2$$

It follows that $(a_0, a_1, a_2, a_3, a_4)$ is one of

$$(d, 3+d, d, d, 2+d), (d, -3+d, d, d, -2+d)$$

for some integer d. Plugging in the former yields that

$$5d^2 + 10d = d^2 + 2(0+3+0+0+2)d = 175 \implies d = 5$$

Similarly, the latter yields

$$5d^2 - 10d = 175 \implies d = 7$$

Then $(a_0, a_1, a_2, a_3, a_4)$ is one of

$$(5, 8, 5, 5, 7), (7, 4, 7, 7, 5)$$

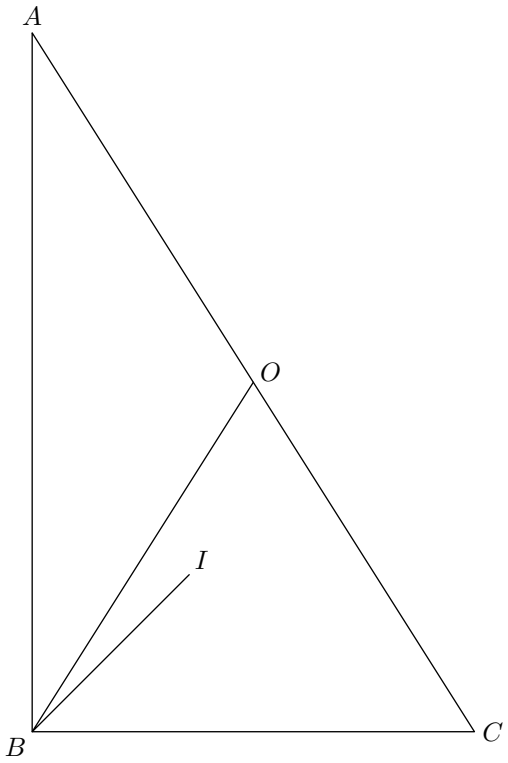
Then the desired value is 24.

Geometry

1. Consider right triangle ABC with $\angle B = 90^\circ$, with incenter I and circumcenter O, satisfying $\angle IBO = 20^\circ$. Compute the absolute difference between $\angle A$ and $\angle C$.

Proposed by Aaryan Vaishya

Answer: 40

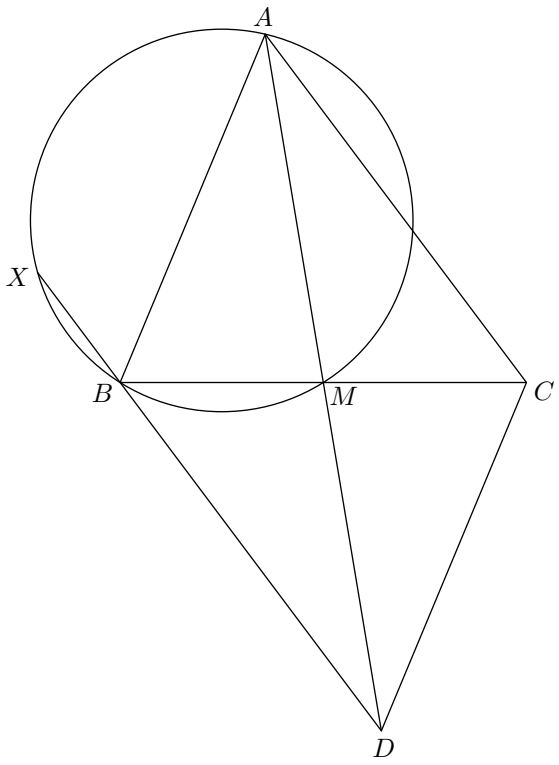


Without loss of generality, assume $\angle A < \angle C$. Then we can compute $\angle OBC = 90^\circ - \angle BAC > \angle IBC = 45^\circ$, so $20^\circ = \angle OBI = \angle OBC - \angle IBC = 45^\circ - \angle BAC$, so $\angle BAC = 25^\circ, \angle BCA = 65^\circ$, giving the desired as 40°

2. Consider triangle ABC, satisfying $AB = 13, BC = 14, AC = 15$. Let M be the midpoint of BC. Let the line parallel to AC through B intersect (ABM) at X. Compute BX.

Proposed by Aaryan Vaishya

Answer: $\frac{71}{15}$



Let D be the reflection of A over M. Then $BD \parallel AC$ and so X lies on BD. Considering the power of D with respect to (ABM), we have

$$DB \cdot DX = DM \cdot MA = 2(AM^2)$$

We have

$$AM^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

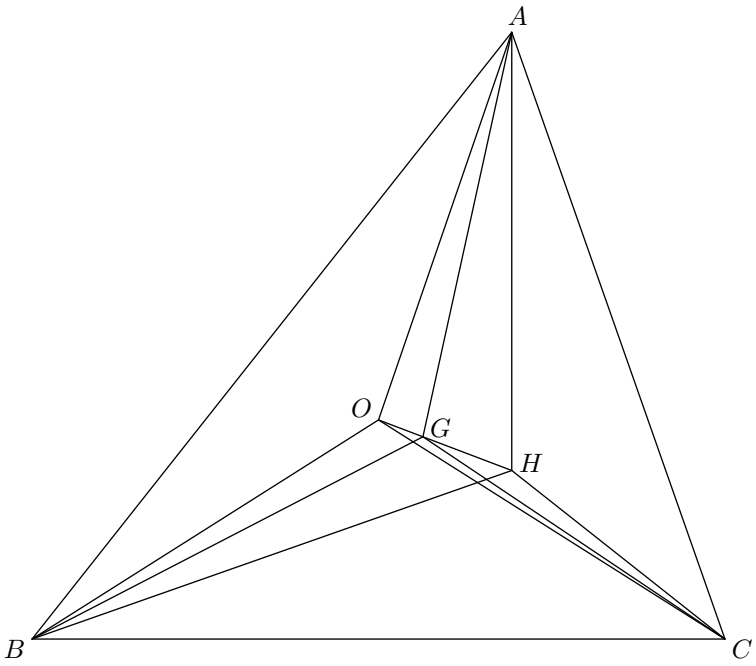
Then

$$BX = \frac{2(AM^2)}{DB} - DB = \frac{2b^2 + 2c^2 - a^2}{2b} - b = \frac{2c^2 - a^2}{2b} = \frac{71}{15}$$

3. Consider acute triangle A, B, C with orthocenter H and circumcenter O. Given that $[BCH] = 10, [BCO] = 13, [ACO] = 14$, compute $[ABH]$.

Proposed by Aaryan Vaishya

Answer: 18

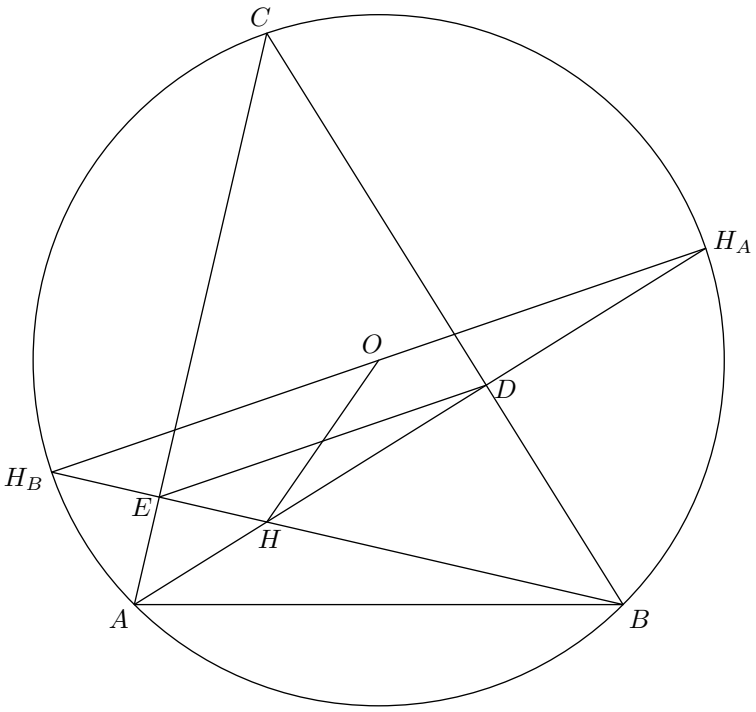


Consider the centroid G. Since $OG : GH = 1 : 2$, we have $2[BCO] + [BCH] = 3[BCG]$ (drop altitudes and use weighted average), which gives $[BCG] = 12$, and thus $[ABC] = 36$. Then $[ABO] = 36 - 13 - 14 = 9$, $[ABH] = 3[ABG] - 2[ABO] = 36 - 18 = 18$.

4. Consider triangle ABC with circumcenter O and orthocenter H. The A-altitude meets BC at D and B-altitude meets AC at point E. Given that DE bisects OH, $AB = 20$, $AC = 24$, compute BC.

Proposed by Tom Zhang

Answer: $12\sqrt{2} + 4\sqrt{7}, 12\sqrt{2} - 4\sqrt{7}$



Consider the homothety centered at H with scale factor 2. It follows from the Orthocenter Reflection Lemma that the images of D and E under this homothety are the reflections of H over BC and AC respectively. Denote these images by H_A and H_B respectively. We now have that H_AH_B passes through O and so is a diameter of the circumcircle of ABC. We have

$$\measuredangle H_ACH_B = \measuredangle H_AAB + \measuredangle BCA + \measuredangle ACH_B = \measuredangle BCH + \measuredangle BCA + \measuredangle HCA = 2\measuredangle BCA$$

in directed angles. Then

$$2\measuredangle BCA = \measuredangle H_ACH_B = 90$$

and so $\angle ACB$ is either 45 or 135. We cannot have $\angle ACB = 135$, since $AC > AB$. Then $\angle ACB = 45$, and so by the Law of Cosines, we have, letting $x = BC$, that

$$24^2 + x^2 - 2 \cdot \frac{\sqrt{2}}{2} \cdot 24 \cdot x = 20^2$$

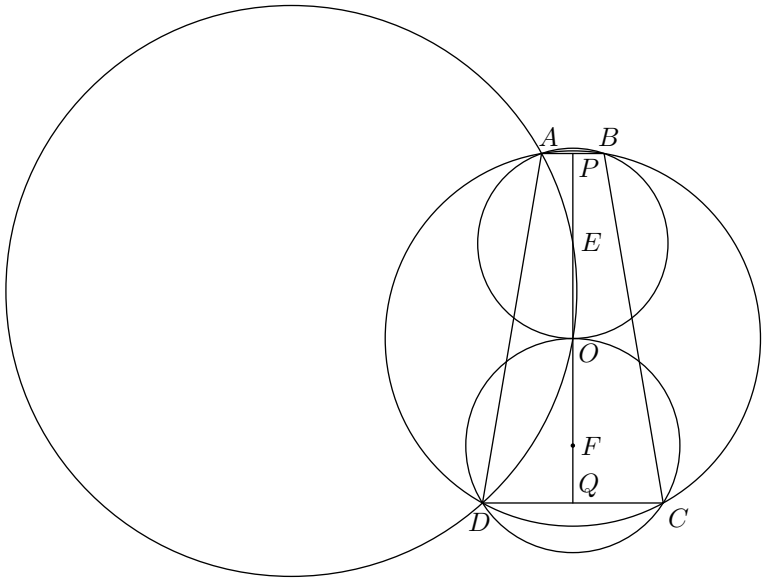
Both of the solutions to this quadratic yield valid triangles, and they are

$$12\sqrt{2} + 4\sqrt{7}, 12\sqrt{2} - 4\sqrt{7}$$

5. Consider quadrilateral $ABCD$ with circumcircle centered at O. Given $AB = 2 < CD$, the circumcircle of $\triangle ABO$ is tangent to the circumcircle of $\triangle COD$. The circumcircle of $\triangle AOD$ passes through the center of the circumcircle of $\triangle ABO$. Given that the radius of the circumcircle of $ABCD$ is 6, compute the area of $ABCD$.

Proposed by Tom Zhang

Answer: $\frac{595\sqrt{35}}{81}$



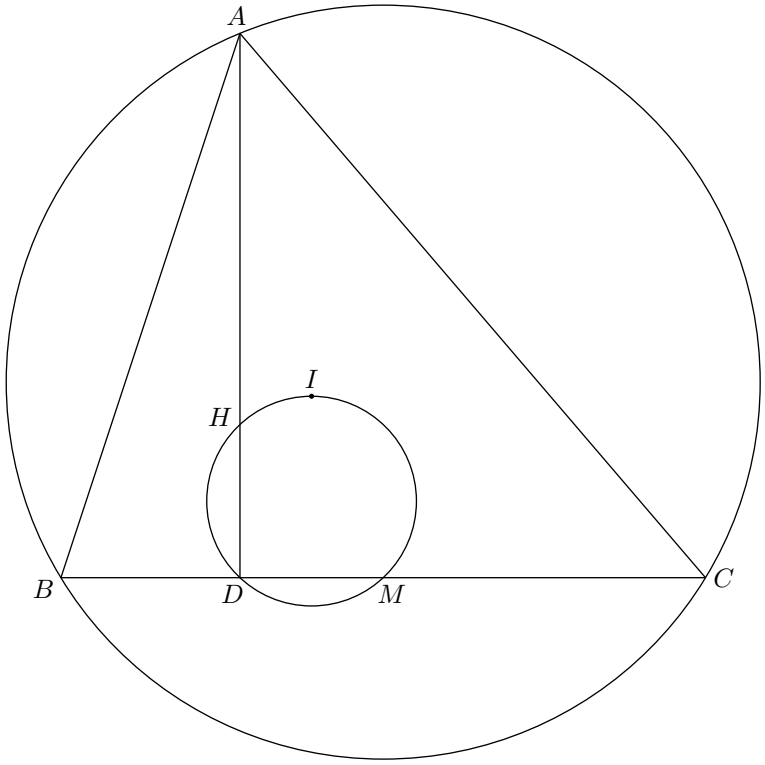
Denote the center of the circumcircle of $\triangle ABO$ as E, the center of (OCD) is F. Since two circles are tangent at O; E, O, F are colinear implies line EF is perpendicular to both AB, CD, $AB||CD$, which means $ABCD$ is an isosceles trapezoid. Let line EF meet AB, CD at P, Q

Connect ED, $\angle AOE = \angle ODE = \frac{\angle AEP}{2} = \frac{\angle ODA}{2} = \frac{\angle OAD}{2}$, it indicates $\angle DOQ = \angle OAE = 3\angle AOE$. Since $AB = 2, AO = BO = 6, OP = \sqrt{35}$, we have $\sin\angle AOE = \frac{1}{6}, \sin\angle DOQ = \frac{13}{27}$ by triple angle formula, $DQ = 6 \cdot \frac{13}{27} = \frac{26}{9}. CD = \frac{52}{9}, OQ = \frac{8\sqrt{35}}{9}$ the area is $\frac{17\sqrt{35}}{9} \cdot \frac{70}{9} \cdot \frac{1}{2} = \frac{595\sqrt{35}}{81}$.

6. Consider acute triangle ABC satisfying $AB = 8, AC = 10$. Given that the orthocenter of ABC, the incenter of ABC, the midpoint of BC, and the foot of the A-altitude are concyclic, compute BC.

Proposed by Aaryan Vaishya

Answer: 9



Define H, D, M, I as standard. Define M_a to be the midpoint of arc BC, and I_a to be the A-excenter.

Let $f(P)$ for a point P denote the difference of the power of P with respect to (HDM) and the power of P with respect to (ABC). Since f is linear, we can write $af(A) + bf(B) + cf(C) = (a+b+c)f(I)$, which gives $BC \cdot 2R \sin B \sin C \cdot 2R \cdot A + AC \cdot \frac{BC}{2} \cdot AB \cos B + AB \cdot \frac{BC}{2} \cdot AC \cos C = 2s \cdot 2Rr$. We can simplify the left side as $BC \cdot AB \cdot AC(\cos A + \frac{\cos B}{2} + \frac{\cos C}{2})$ We can simplify the right side as $4Rrs = 4R[ABC] = AB \cdot BC \cdot AC$. Dividing this from both sides gives $\cos A + \frac{\cos B + \cos C}{2} = 1$. We can simplify the left side using sum and product as $\frac{\cos A + \cos B}{2} + \frac{\cos A + \cos C}{2} = \sin \frac{C}{2}(\cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2}) + \sin \frac{B}{2}(\cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{B}{2}) = 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + \cos \frac{A}{2}(\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2}) = \frac{r}{2R} + \cos^2 \frac{A}{2} = \frac{r}{2R} + \frac{1+\cos A}{2}$. Thus we have $\frac{r}{2R} + \frac{1+\cos A}{2} = 1$. Multiplying both sides by $2R$ yields $r + R \cos A = R$. Using half angle again yiels $r = 2R \sin^2 \frac{A}{2}$, or $\frac{r}{\sin \frac{A}{2}} = 2R \sin \frac{A}{2}$, or $AI = BM_a = IM_a$. Of course, this implies that $AI_a = 3AI$, thus $\frac{s}{s-a} = 3$, so $3a = 2s$, or $2a = b + c$. Thus $a = \frac{8+10}{2} = 9$.

Combinatorics

1. Yusuf and Darsh are playing a game. First, Darsh rolls a die to receive a random number between 1 and 6 inclusive. Then, Yusuf rolls his rigged die that gives a random number between Darsh's number and 6 inclusive. Compute the expected value of the product of their numbers.

Proposed by Aaryan Vaishya

Answer: $\frac{217}{12}$

We sum over Darsh's 6 possible rolls, which each happen with probability $\frac{1}{6}$. The expected value of Yusuf's roll if Darsh rolls an n is $\frac{n+6}{2}$, so we are looking to find

$$\sum_{n=1}^{6} \frac{1}{6} (n) \left(\frac{6+n}{2} \right),$$

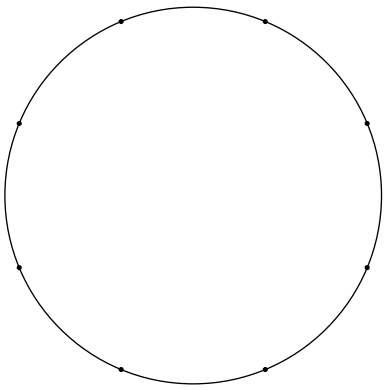
which is easily computed to be $\frac{217}{12}$.

2. Compute the probability that 4 chords each formed from 2 randomly chosen points on a circle intersect at exactly 4 points and form a convex, non-self-intersecting quadrilateral.

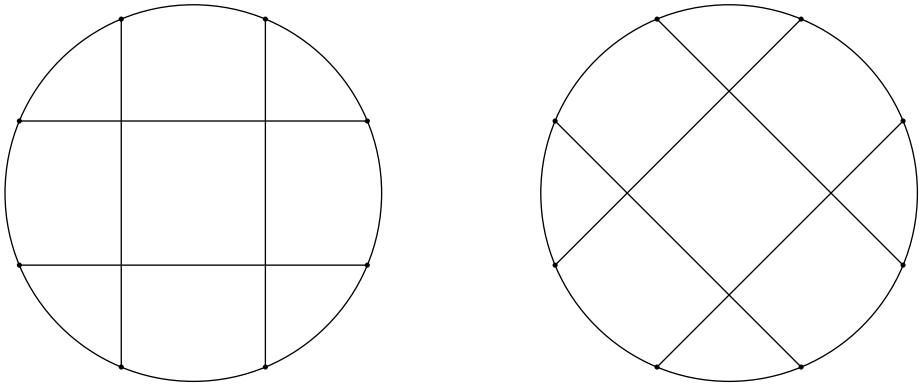
Proposed by Aaryan Vaishya

Answer: $\frac{2}{105}$

We fix the 8 points on the circle and draw the chords from them.



There are a total of $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!} = 105$ ways to draw the chords. We claim the only ways to form a quadrilateral from exactly 4 points are the two shown below

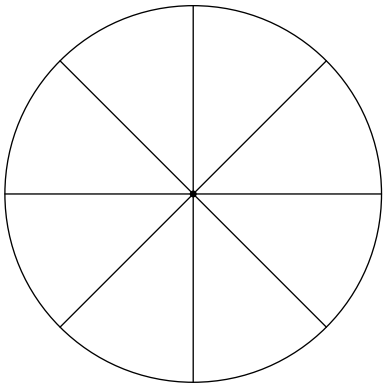


This is true by considering any quadrilateral in the interior of the circle. The only way for it to be formed from 8 points is by extending each side until it intersects the circumference of the circle, giving our 4 chords that way. It's not hard to see that this will generate a configuration like the ones shown above. Thus, the answer is $\frac{2}{105}$.

3. Aaryan is cheating in darts. He takes a dartboard with 8 sectors separated by lines and removes 4 of the lines randomly. Compute the expected number of sectors after he does this.

Proposed by Aaryan Vaishya

Answer: 4



Each move removes exactly one sector, so the answer is always 4.

4. Compute the number of permutations of $\{1,2,3,4,5\}$ such that the absolute difference between any pair of elements does not equal the absolute difference between their indices.

Proposed by Aaryan Vaishya

Answer: 10

We separate into cases based on where the 3 is. If it is on the edge, without loss of generality let it be at the first position. Then second position then can either be 1 or 5, without loss of generality let it be 1. The third position then must be 4, because 2 cannot be next to 1 and 5 cannot be 2 away from 3. The fourth position must then be 2, as 4 and 5 cannot be next to each other. The complete permutation is then 31425, which works. We have to multiply by 4 for symmetry, as we could've chosen to put 3 on either end and we could've chosen the next element to be one of two things, giving us 4 working in permutations in this case. If 3 is not at the edge, it suffices to consider the $3 \cdot 2 \cdot 2$ permutations that have no adjacent elements differing by 1 (the number comes from arranging 135 as a block and 2, 4). Listing gives

$$13524, 13542, 53124, 53142, 21354, 41352, 25314, 45312, 24135, 42135, 24531, 42531$$

We can eliminate

$$13542, 53124, 21354, 45312, 42135, 24531$$

as all of these either have 4, 1 or 5, 2 located two apart. The remaining permutations can easily be inspected to work, as 1, 5 are two away in all of them so no difference of 4 can ruin us, we manually inspect for differences of 3, no differences of 2 can kill us since 2, 4 are either on opposite sides or adjacent, and 1, 3, 5 are all consecutive, and obviously no adjacent difference 1 pairs exist. Thus there are $12 - 6 = 6$ cases that work. The total is then $4 + 6 = 10$.

5. Positive real numbers x, y, z satisfying $x + y + z = 1$ are chosen randomly and uniformly. Compute the probability that

$$169yz + 196zx + 225xy \geq 64x + 49y + 36z$$

Proposed by Xuzhou Ren, Aaryan Vaishya, Yusuf Sheikh

Answer: $\frac{4\pi}{21}$

The key idea is that x, y, z can be interpreted as a barycentric coordinate in a 13-14-15 triangle. The problem then asks if a randomly chosen barycentric coordinate lies in the incenter of of the triangle. We can show that the probability that a randomly chosen barycentric coordinate lying in a region is proportional to the area of this region, the proof is obvious for equilateral triangles by considering the surface $x + y + z = 1$ and we can easily linear transform the equilateral triangle into whatever we want, giving the desired result. Thus the answer is just $\frac{\pi \cdot 4^2}{84} = \frac{4\pi}{21}$

6. Four hooligans, Tiger, Darsh, Andrew, and Aaryan, live on an archipelago of 20 islands connected by one-way bridges. Yusuf places the bridges so that each island has a unique bridge leading towards it and a unique bridge leading away from it, with each possible placement of bridges being equally likely, and then randomly places the hooligans on different islands. A group of hooligans can have a *tea party* if they can, after traversing bridges, end up on the same island. Compute the probability that the four hooligans cannot have a tea party, but if Aaryan is excluded, then the remaining three can.

Proposed by Aaryan Vaishya

Answer: $\frac{1}{12}$

The key idea is to consider the bijection from permutations in cycle notation (permutations written in order of cycles, which are ordered descending by smallest element, as an example consider writing 5746132 as $(346)(27)(15)$) to permutations by dropping the brackets. It is easy to see that this function is injective, and since there are $n!$ permutations in cycle notation as well as $n!$ permutations overall, the function is a bijection.

The probability that $1, 2, 3$ are in a cycle but 4 is not is the probability that $2, 3$ come after 1 but 4 doesn't in a randomly selected permutation, which can then be inversely mapped through the bijection to a permutation where the given cycle condition is satisfied. Obviously, this happens with probability $\frac{2}{24} = \frac{1}{12}$