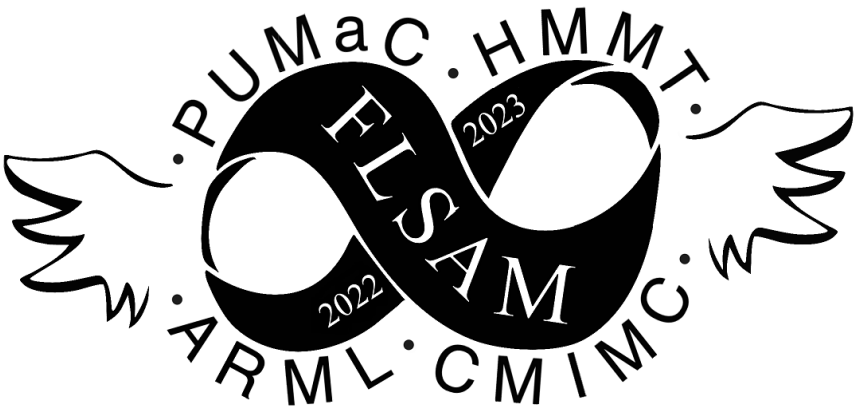
Florida Student Association of Mathematics



2024 Intro Meeting Test

October 2024

Welcome to the **2024 Intro Meeting Test**! This is the individual round, consisting of 10 problems in the subjects of algebra, geometry, number theory, and combinatorics. Each problem is worth 1 point, and problems are not necessarily in order of increasing difficulty.

*You will have **60 minutes** to complete the test. Good luck, and have fun!*

Answer Key

1. 50
2. -14
3. 1764
4. 1800
5. 30
6. 486
7. -4
8. 635
9. 2331
10. 73

1. Jason writes all the positive integers from 1 to 36, inclusive, on a blackboard. Daniel then erases two of his favorite distinct numbers. Afterwards, Jeffrey observes that the sum of the remaining numbers is equal to the product of the two numbers that Daniel erased. Determine the sum of the two numbers that Daniel erased.

Answer: 50

Proposed by Jacob Khohayting

Let the two numbers be x, y. The sum of the original numbers was $\frac{1}{2} \cdot 36 \cdot 37 = 666$, so we have $666 - x - y = xy$, rearranging and using SFFT gives $667 = (x+1)(y+1)$. Since $667 = 23 \cdot 29$, we have $x = 22, y = 28$, giving the answer as $\boxed{50}$.

2. Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the solutions to the equation $x^7 - x + 2 = 0$. Compute $x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7 + x_7^7$.

Answer: -14

Proposed by Aaryan Vaishya

We know $x_i^7 = x - 2$, so summing gives the desired as $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 14$, by Vieta's we know the answer is $\boxed{\text{-14}}$.

3. A *geonacci* sequence is a sequence defined by the rule $a_{n+2} = a_{n+1}a_n$. Given $a_1 = 3, a_2 = 10$, compute the number of factors of a_8.

Answer: 1764

Proposed by Roger He

We track the powers of $2, 3, 5$ in the prime factorization of a_i. We can track the power of 2 and 5 together, we get $0, 1, 1, 2, 3, 5, 8, 13$. We can track the power of 3 as $1, 0, 1, 1, 2, 3, 5, 8$. We thus have $a_8 = 2^{13}3^8 5^{13}$, which has $9 \cdot 14 \cdot 14 = 1764$ divisors.

4. Sasha's favorite number satisfied the following conditions:

- Exactly 4 distinct divisors of her favorite number have 4 factors.
- Exactly 6 distinct divisors of her favorite number have 6 factors.

Find the minimum possible value of Sasha's favorite number.

Answer: 1800

Proposed by Jacob Khohayting

If the number has two distinct prime divisors p, q, we see that the maximal number of divisors that have 4 factors is just 3 (p^3, q^3, pq), so we can rule this case out. If the number has four or more distinct prime divisors, we can see that the product of every pair of prime divisors gives a divisor that has 4 factors, since $\binom{4}{2} > 4$ we can rule this case out. Thus the number has three distinct prime divisors p, q, r. From here, we can easily see that we can get 6 divisors with 6 factors by using the numbers of the form p^2q, where p, q are symmetrically substituted with the other prime divisors. To get 4 divisors with 4 factors, we require one of the primes to It is optimal to have a number of the form $p^3q^2r^2$ for distinct primes p, q, r, so setting $p = 2, q = 3, r = 5$ gives the answer of $\boxed{1800}$

5. Given that $x^3 - ax^2 + bx - b = 0$ has 3 positive integer solutions, and that a and b are positive integers, compute the sum of all possible values of a.

Answer: 30

Proposed by Luis Fonseca

Let the roots of the polynomial be p, q, r. Then using Vieta, we have $pq + qr + pr = pqr$, so $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. We desire to find solutions to this Diophantine. This is a rather standard problem, it suffices to do casework on p, q, r. Assume $p \geq q \geq r$. If $p > 3$, we have no solutions since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ is bounded by $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < 1$. If $p = 3$, we desire $\frac{1}{q} + \frac{1}{r} = \frac{2}{3}$, by size we must have $q = r = 3$, giving $(3, 3, 3)$ as a solution. If $p = 2$, we must have $q \leq 4$ otherwise the whole expression is less than 1. Testing, we see $(2, 3, 6)$ and $(2, 4, 4)$ are both valid solutions. The corresponding values of a in these cases are just the sum of p, q, r, giving the possible values as $2 + 3 + 6 = 11, 2 + 4 + 4 = 10, 3 + 3 + 3 = 9$, so the answer is $\boxed{30}$.

6. Find the number of permutations of the word "FLORIDA" such that each letter appears at most two spots later than it does in FLORIDA.

Answer: 486

Proposed by Shrey Gupta

There are 3 ways to place the F, then there are 4 positions we can place the L but one is taken by the F, giving 3 ways to place the L, then there are 5 positions to place the O but two are taken by the previous letters, giving 3 ways to place the O, then there are 6 positions to place the R but three are taken by previous letters, giving 3 ways to place the R, then there are 7 positions to place the I but four are taken by previous letters, then there are 7 positions to place the D but five are taken by previous letters, giving 2 ways to place the D, finally there is just one way to place the A. The answer is then $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = \boxed{486}$.

7. Suppose a, b, and c are the solutions to $P(x) = x^3 + 4x^2 - 1 = 0$. Compute $\left(a + b + \frac{1}{c}\right)\left(b + c + \frac{1}{a}\right)\left(c + a + \frac{1}{b}\right)$.

Answer: -4

Proposed by Jayden Lang

Notice $\left(a + b + \frac{1}{c}\right)\left(b + c + \frac{1}{a}\right)\left(c + a + \frac{1}{b}\right) = (a + b + ab)(b + c + bc)(c + a + ca) = (a + b - bc - ac)(b + c - ac - ab)(c + a - ab - bc) = (a + b)(1 - c)(b + c)(1 - a)(a + c)(1 - b) = P(-4) \cdot P(1)$.

We have that $P(-4) = -64 + 64 - 1 = -1$ and $P(1) = 1 + 4 - 1 = 4$, so the answer is $\boxed{-4}$.

8. Consider $\triangle ABC$ with $AB = 13, BC = 14, AC = 15$. Erect squares $ABDE, BCFG, ACHI$ outside the triangle, denote that the centers of the squares above as X, Y, Z, the area of $\triangle XYZ$ can be written in the simplest form of $\frac{m}{n}$ when $\gcd(m, n) = 1$ and $m, n \in \mathbb{N}$. Find $m + n$.

Answer: 635

Proposed by Jingyi Li and Yibo Zhang

Let $a = 13, b = 14, c = 15$. So we have $[XYZ] = [ABC] + [AXB] + [BYC] + [AZC] - [AXZ] - [BXY] - [CYZ]$.

Note $[AXB] + [BYC] + [AZC] = \frac{a^2 - b^2 + c^2}{8}$, $[AXZ] = \frac{1}{2} \cdot \frac{a}{\sqrt{2}} \cdot \frac{c}{\sqrt{2}} \sin(90^\circ + \angle BAC) = \frac{ac\cos(\angle BAC)}{4}$. By Law of Cosines, $\cos(\angle BAC) = \frac{a^2 - b^2 + c^2}{2ac}$, we have $[AXZ] = \frac{a^2 - b^2 + c^2}{8}$. Similarly, we can conclude $[AXZ] + [BXY] + [CYZ] = \frac{a^2 + b^2 + c^2}{8}$. Now, we have $[XYZ] = \frac{a^2 + b^2 + c^2}{8} + [ABC] = \frac{631}{4}$. Hence, $m + n = \boxed{635}$.

9. Find the number of ordered triples (a, b, c) of integers such that

$$a \leq b + c \leq a + 20$$
$$b \leq c + a \leq b + 20$$
$$c \leq a + b \leq c + 20$$

Answer: 2331

Proposed by Darsh Patel

Equivalently, we have

$$0 \leq -a+b+c \leq 20$$
$$0 \leq a-b+c \leq 20$$
$$0 \leq a+b-c \leq 20$$

Let

$$x = -a+b+c$$
$$y = a-b+c$$
$$z = a+b-c$$

Then

$$a = \frac{y+z}{2}$$
$$b = \frac{z+x}{2}$$
$$c = \frac{x+y}{2}$$

It is now equivalent to determine how many (x, y, z) are such that $0 \leq x, y, z \leq 20$ and $\frac{y+z}{2}, \frac{z+x}{2}, \frac{x+y}{2} \in \mathbb{Z}$. However, $\frac{y+z}{2}$ is an integer if and only if $y \equiv z \pmod 2$, and so the second condition is equivalent to $x \equiv y \equiv z \pmod 2$. Then considering when x, y, and z are all even or all odd, there are $11^3 + 10^3 = \boxed{2331}$ possibilities.

10. Consider a parabola with focus F and directrix ℓ. A circle ω with center O passes through F and intersects ℓ at A_1 and A_2, and segment $\overline{FO}$ intersects the parabola at B. Given $FA_1 = 5$, $FA_2 = 7$, $A_1A_2 = 6$, FB can be expressed in the simplest form of $\frac{a\sqrt{b}}{c}$ where $\gcd(a, c) = 1$ and b is square-free. Find $a + b + c$.

Answer: 73

Proposed by Ashvin Sinha

B is the center of the circle tangent to ω and the directrix, as any point on the parabola is equidistant from the directrix and F. Call D the foot of the bisector of $\angle A_1FA_2$ and call M the midpoint of arc A_1A_2 on ω. Then, $\frac{FB}{FO} = \frac{FD}{FM}$. We can find FD through Stewart's Theorem, giving $FD = \frac{\sqrt{105}}{2}$ aand DM from POP, giving $DM = \frac{\sqrt{105}}{6}$. Thus, our left hand side is equivalent to $\frac{3}{4}$. FO is our circumradius, which is equivalent to $\frac{35\sqrt{6}}{24}$. Multiplying this by $\frac{3}{4}$ gives $\frac{35\sqrt{6}}{32}$ for FB. Thus, our answer is $\boxed{73}$.