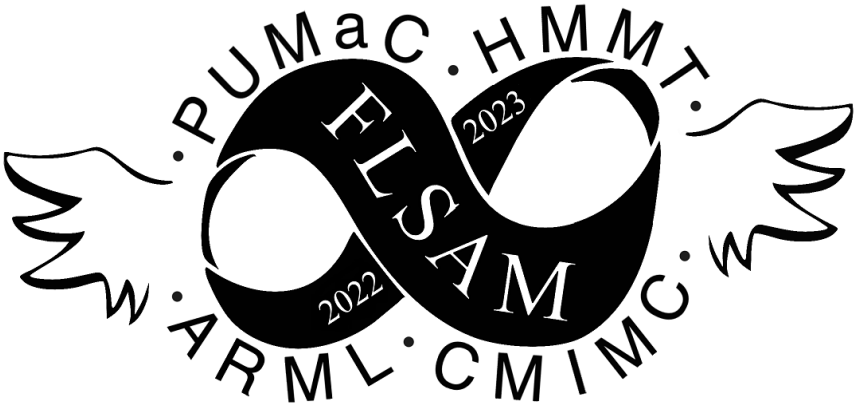
Florida Student Association of Mathematics



2024 PUMaC Tryout

January 21-22, 2023

Solutions

Algebra

1. Consider polynomial $x^3 + kx^2 - 40x + 71$ with roots r, s, t. Compute the minimum value of $(r-s)^2 + (s-t)^2 + (t-r)^2$.

Answer: 240

Proposed by Aaryan Vaishya

Expanding our expression gives $2r^2 + 2s^2 + 2t^2 - 2rs - 2ts - 2tr$. By Vieta's, we know that $rs + ts + tr = -40$ and $r + s + t = -k$. Since $r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs + ts + tr)$, our expression simplifies to $2k^2 + 240$. Thus, to minimize, $k = 0$ and our answer is $\boxed{240}$.

2. Compute

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 3}{n(n+1)(n+2)(n+3)}.$$

Answer: $\frac{7}{12}$

Proposed by Xuzhou Ren

Rewrite

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 3}{n(n+1)(n+2)(n+3)} = \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+3)} + \frac{3}{n(n+1)(n+2)(n+3)} \right).$$

The first sum telescopes to $\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right)$ and the second sum is well-known to be $\frac{3}{3 \cdot 3!}$, so the answer is $\boxed{\frac{7}{12}}$.

3. Compute the least real number r such that

$$\prod_{i=0}^{n} (2\cos(2^i\theta) - 1) \leq r$$

over all $0 < \theta < \arccos\left(\frac{1}{2024}\right)$ and positive integers n.

Answer: $\frac{3036}{1013}$

Proposed by Darsh Patel

Looking for a telescope, we find

$$\begin{aligned} &2\cos(2\phi) + 1 \\ =&2(2\cos^2(\phi) - 1) + 1 \\ =&4\cos^2(\phi) - 1 \\ =&(2\cos(\phi) + 1)(2\cos(\phi) - 1) \end{aligned}$$

Then

$$(2\cos(\theta) + 1)\left(\prod_{i=0}^{n} (2\cos(2^i\theta) - 1)\right) = 2\cos(2^{n+1}\theta) + 1$$

and

$$\prod_{i=0}^{n} (2\cos(2^i\theta) - 1) = \frac{2\cos(2^{n+1}\theta) + 1}{2\cos(\theta) + 1}$$

Then

$$\frac{2\cos(2^{n+1}\theta)+1}{2\cos(\theta)+1} \leq \frac{3}{2\cos(\theta)+1} < \frac{3}{2\left(\frac{1}{2024}\right)+1} = \frac{3036}{1013}$$

and so $r = \boxed{\frac{3036}{1013}}$.

To show that this r works, we need to show that over all θ and n we can get arbitrarily close to r. For a fixed θ, we will try to pick n so that $2^{n+1}\theta$ is arbitrarily close to $2k\pi$ for some positive integer k, in which case we can achieve a value arbitrarily close to

$$\frac{3}{2\cos(\theta)+1}$$

We need that $2^{n+1}\left(\frac{\theta}{2\pi}\right)$ can be arbitrarily close to an integer. This can be achieved by picking θ such that $\frac{\theta}{2\pi}$ terminates in binary, in which case $2^{n+1}\left(\frac{\theta}{2\pi}\right)$ can be an integer. We can pick θ so that the binary expansion of $\frac{\theta}{2\pi}$ is a truncated version of the binary expansion of $\frac{\arccos\left(\frac{1}{2024}\right)}{2\pi}$. Further, since such values of θ are arbitrarily close to $\arccos\left(\frac{1}{2024}\right)$, we can get values arbitrarily close to $\frac{3}{2\left(\frac{1}{2024}\right)+1}$, as desired.

4. Let n be the number of polynomials $P(x)$ with complex coefficients such that $\deg(P) < 11$ and

$$x^{12} - x \mid P(x)^{12} - P(x).$$

Compute the last three digits of n.

Answer: 812

Proposed by Xuzhou Ren

The problem statements reduces to the fact that P must map 0 and the 11th roots of unity to 0 or an 11th root of unity. Interpolating the leading coefficient then gives

$$\frac{P(0)}{(0-1)(0-z)\cdots(0-z^{10})} + \sum_{i=0}^{10} \frac{P(z^i)}{z^i \prod_{i\neq j}(z^i - z^j)}.$$

Now we introduce a key claim.

Let p be a prime. Then, if the sum of p pth roots of unity is 0, then they must all be distinct.

Let the p roots of unity are $z^{a_1}, z^{a_2}, \ldots, z^{a_p}$ for $0 \leq a_1, \ldots, a_p < p$. Consider $P(x) = x^{a_1} + x^{a_2} + \cdots + x^{a_p}$. This is an integer polynomial, but z is also a root of this polynomial. Thus, the minimal polynomial of z must divide $P(x)$, i.e.

$$x^{p-1} + \cdots + x + 1 \mid P(x).$$

But $P(1) = p$, so if $P(x) = (x^{p-1} + \cdots + x + 1)Q(x)$, then $Q(1) = 1$. But $\deg(P) \leq p-1$, so either $Q \equiv 0$ or $\deg(P) = p-1$ and Q is constant, i.e. $Q \equiv 1$.

Now, simplifying the interpolating polynomial gives

$$-P(0) + \frac{1}{p}\sum_{i=0}^{p-1} P(z^i) = 0,$$

so we have 3 cases.

Case 1. $P(0) = 0 \implies P(z^i)$ is a permutation by the claim, so there are $11! = 39916800$ solutions in this case.

Case 2. $P(0) = 0 \implies P(z^i) = 0$, trivially 1 solution here.

Case 3. $P(0) = z^k \implies P(z^i) = z^k$ since the real part average of the rightmost term is at most 1, and we can shift P accordingly to get that all the real parts are 1, so there are 11 solutions in this case.

Thus, the final answer is $11! + 11 + 1 = 39916\boxed{812}$.

Number Theory

1. Given that 23 in base a is equal to 34 in base b which is equal to 45 in base c, compute the smallest c.

Answer: 8

Proposed by Aaryan Vaishya

Expressing in terms of a, b, and c gives $2a + 3 = 3b + 4 = 4c + 5 = n$. Thus, we know that n is 1 (mod 4), 1 (mod 3) and 1 (mod 2). So, n must be 1 (mod 12), and we also need $a > 3$, $b > 4$ and $c > 5$. c is the most limiting condition, so n must be greater than 25. The smallest n which satisfies these conditions is 37, and the smallest c is $\boxed{8}$.

2. Compute the number of ordered pairs of integers (x, y) for which

$$\frac{xy}{x+y+1}$$

is an integer between 1 and 16 inclusive.

Answer: 272

Proposed by Darsh Patel

Suppose $\frac{xy}{x+y+1} = n$. Using SFFT, we have

$$\frac{xy}{x+y+1} = n \iff xy = n(x+y+1) \iff (x-n)(y-n) = n(n+1)$$

Excluding the cases with $x + y + 1 = 0$, which are $(-1, 0)$ and $(0, -1)$, this has

$$2\tau(n(n+1)) - 2 = 2\tau(n)\tau(n+1) - 2$$

solutions. The desired value is then

$$\sum_{n=1}^{16} (2\tau(n)\tau(n+1) - 2) = 2\left(\sum_{n=1}^{16} \tau(n)\tau(n+1)\right) - 32$$

n	$\tau(n)$
1	1
2	2
3	2
4	3
5	2
6	4
7	2
8	4
9	3
10	4
11	2
12	6
13	2
14	4
15	4
16	5
17	2

We compute this as $2(152) - 32 = \boxed{272}$.

3. Let $P(x)$ be a degree 4 polynomial with integer coefficients, and let S be the number of integers n with $1 \leq n \leq 5005$ for which 5005 divides $P(n)$. Over all possible such $P(x)$, compute the number of distinct possible values of S.

Answer: 161

Proposed by Darsh Patel

We prime factorize 5005 as $5 \cdot 7 \cdot 11 \cdot 13$. We will find the possible number of solutions to

$$P(n) \equiv 0 \pmod{p}$$

for $p = 5, 7, 11, 13$. It follows from the Chinese Remainder Theorem that the number of solutions to $P(n) \equiv 0 \pmod{5005}$ is the product of these.

We have that $P(n) \equiv 0 \pmod{p}$ has at most 4 solutions if P is nonzero modulo p, since P has degree at most 4. Each of $k = 0, 1, \ldots, 4$ solutions are achievable by taking

$$P(n) \equiv (n-1)(n-2)\ldots(n-k) \pmod{p}$$

Also, this has p solutions if $P(n)$ is 0 modulo p.

It now remains to find the number of S of the form $a_5 a_7 a_{11} a_{13}$, where a_p comes from

$$\{0, 1, 2, 3, 4, p\}$$

We note that each such S works since it follows from the Chinese Remainder Theorem that we can find $P(x)$ having a_p solutions to $P(x) \equiv 0 \pmod{p}$ for each $p = 5, 7, 11, 13$. We count $S = 0$ as 1 such value, and now consider $S \neq 0$. Suppose S is divisible by k of the primes $5, 7, 11, 13$. There are $\binom{4}{k}$ ways to pick these k. If $\nu_3(S) = \ell$, there are $2(4-k-\ell)+1$ possibilities for $\nu_2(S)$. The desired value

is then

$$\begin{aligned}
&\left(\sum_{k=0}^{4}\binom{4}{k}\left(\sum_{\ell=0}^{4-k}(2(4-k-\ell)+1)\right)\right)+1\\
=&\left(\sum_{k=0}^{4}\binom{4}{k}(5-k)^2\right)+1\\
=&160+1\\
=&\boxed{161}
\end{aligned}$$

4. A pair of integers m and n are called *coconspirators* if there exist integers a and b such that 2024 divides both $am+n$ and $m+bn$. Compute the number of pairs (m,n) of nonnegative integers less than 2024 such that m and n are coconspirators.

Answer: 1077670

Proposed by Darsh Patel

We claim that m and n being coconspirators is equivalent to $\gcd(m,2024)=\gcd(n,2024)$. We have that the existence of a with

$$n\equiv -am \pmod{2024}$$

is equivalent to $\gcd(m,2024)\mid\gcd(n,2024)$. Similarly, the existence of b is equivalent to $\gcd(n,2024)\mid\gcd(m,2024)$. Thus, a and b exist if and only if $\gcd(m,2024)=\gcd(n,2024)$.

Now, for $d\mid 2024$, the number of $0\le k<2024$ with $\gcd(k,2024)=d$ is $\varphi\left(\frac{2024}{d}\right)$. So, the desired value is

$$\sum_{d\mid 2024}\varphi\left(\frac{2024}{d}\right)^2=\sum_{d\mid 2024}\varphi(d)^2$$

However, φ is multiplicative, so this is just

$$(\varphi(1)^2+\varphi(2)^2+\varphi(4)^2+\varphi(8)^2)(\varphi(1)^2+\varphi(11)^2)(\varphi(1)^2+\varphi(23)^2)$$

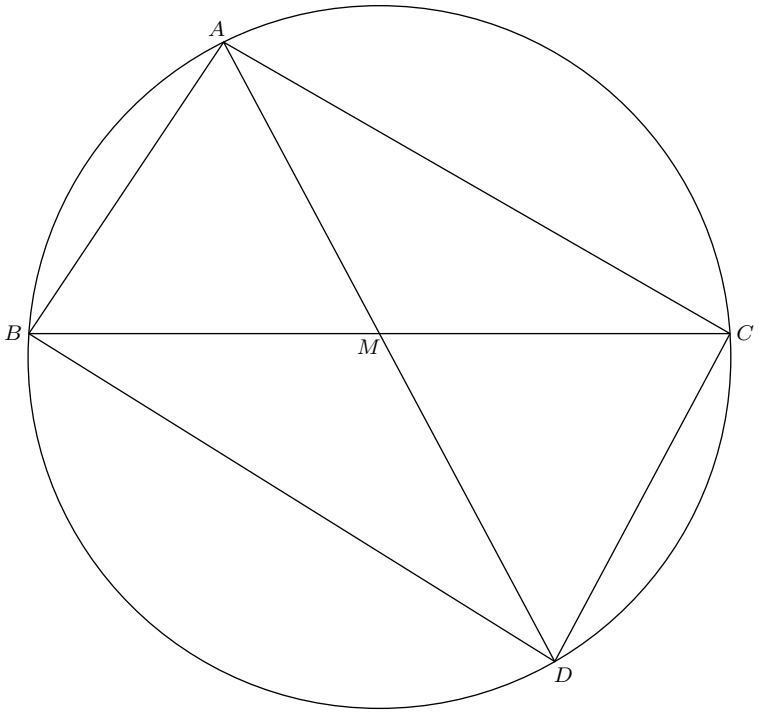
which we compute as $\boxed{1077670}$.

Geometry

1. Consider triangle ABC with $BC=12, AB=6, CA=10$. Let M be the midpoint of BC. Let AM meet the circumcircle of ABC at D. Compute the area of BCD.

Answer: $9\sqrt{14}$

Proposed by Aaryan Vaishya

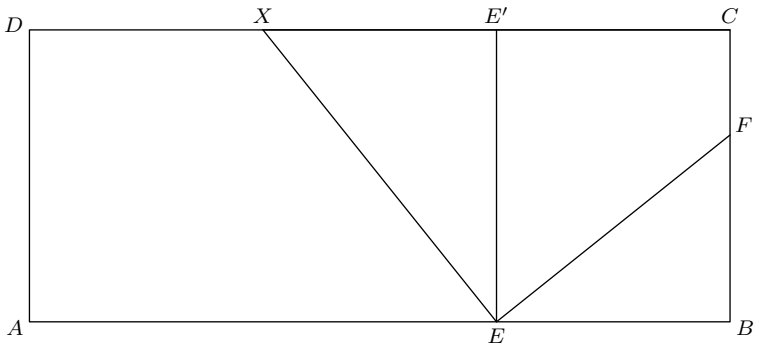


First, we compute the length AM by Appolonius theorem to be $\sqrt{\frac{1}{4}(2(6^2+10^2)-12^2)} = 4\sqrt{2}$. Noting that BAM is isosceles, we can compute $\cos BMA = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}, \sin BMA = \frac{\sqrt{7}}{3}$. Noting that $BM \cdot CM = AM \cdot DM$, we have $DM = \frac{9\sqrt{2}}{2}$, so the desired area is $\frac{1}{2} \cdot \frac{\sqrt{7}}{3} \cdot \frac{1}{2} \cdot 9\sqrt{2} \cdot 12 = \boxed{9\sqrt{14}}$.

2. Rectangle $ABCD$ has $AB = 12$ and $BC = 5$. Line ℓ splits the rectangle into two parts with equal area and meets AB at E. A line perpendicular to line ℓ that goes through E meets BC at F, compute the maximum possible area of EBF.

Answer: $\frac{32}{5}$ or $\frac{864}{5}$

Proposed by Xuzhou Ren



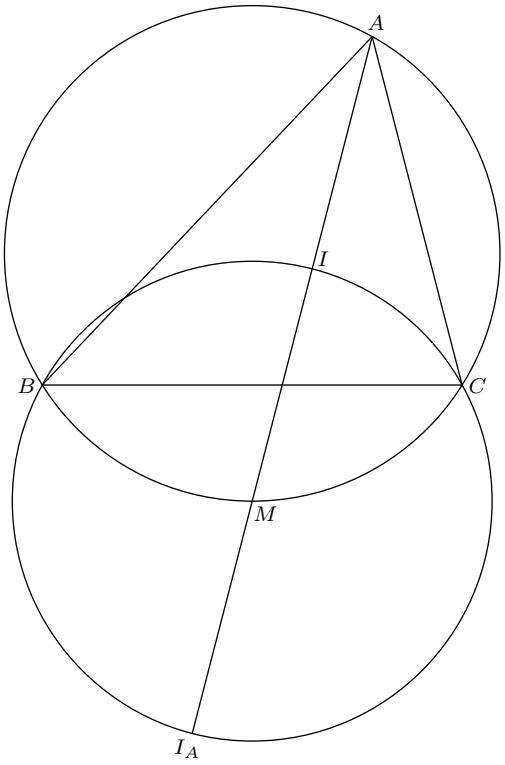
Intended: Let line ℓ meet CD at X and let the projection of E onto CD be E', and let $XE' = x$. Then, since $XE'E$ and FBE are similar, and $EB = 6 - \frac{x}{2}$ and $BF = \frac{x}{5} \cdot \left(6 - \frac{x}{2}\right)$, and the area is $\frac{1}{2} \cdot \frac{x}{5} \left(6 - \frac{x}{2}\right)^2 \leq \boxed{\frac{32}{5}}$ by AMGM.

Also Accepted: Because BC is not clearly defined as a segment, we can maximize the area of EBF by assuming that ℓ goes through A, so A and E are the same point. Since EBF and ABC are now similar, we can find side length BF as $\frac{144}{5}$. This gives our final area of EBF as $\boxed{\frac{864}{5}}$.

3. Consider triangle ABC with $AB = 6$, $AC = 8$. Let M be the midpoint of minor arc BC, and let I be the incenter. Given $AM = 8$, compute AI.

Answer: 4

Proposed by Aaryan Vaishya

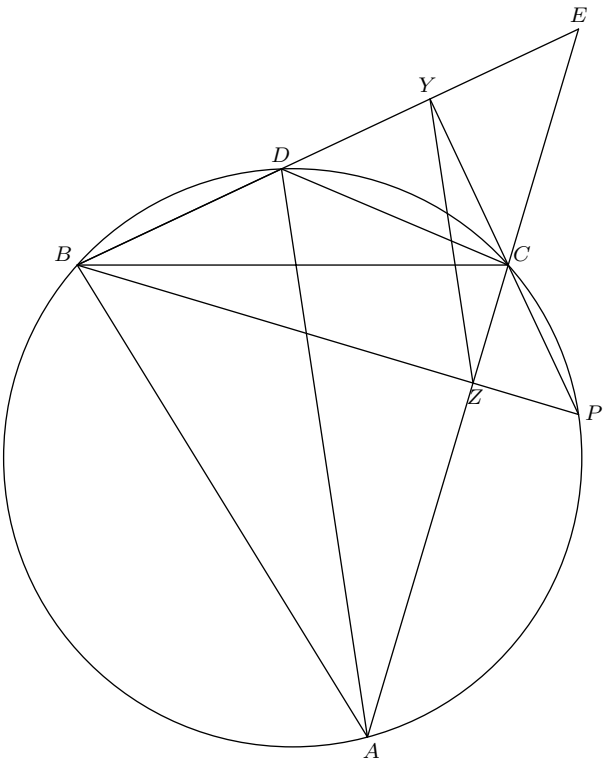


Introduce I_A, the A-excenter. We know that $AI \cdot AI_A = AB \cdot AC = 48, AI + AI_A = 16$ by standard configurations. Solving the resulting quadratic gives gives $AI = 4, AI_A = 12$, since $AI < AI_A$, and the answer is then $\boxed{4}$.

4. Points A, B, C, D lie on a circle. AC and BD intersect at E. Given that the reflection of E over BC is concyclic with $ABCD$, $AB = 4, BC = 7, CD = 8$, compute DA.

Answer: $\frac{217}{32}$

Proposed by Aaryan Vaishya



Letting the line connecting E and its reflection over BC intersect the circle again at P, it follows from the Orthocenter Reflection Lemma that E is the orthocenter of PBC. It now also follows that A and B are the reflections of E over BP and CP respectively. Thus, $BE = 4$ and $CE = 8$, and we already have $BC = 7$. Let Y and Z be the feet of the B and C altitudes respectively in PBC. In order to find DA, we will compute YZ, since $DA = 2(YZ)$.

We direct lengths along lines BE and CE so that the lengths BE and CE are positive. Let $x = EZ$ and $y = |BZ|$. Since $ECY \sim EBZ$ we have $EY = 2x$ and $|CY| = 2y$. Then from Pythagorean Theorem on BYC and BZC we get

$$(2x+4)^2 + (2y)^2 = 49$$
$$(x+8)^2 + y^2 = 49$$

Solving for x, we obtain $x = -\frac{31}{16}$. Then since $BEC \sim ZEY$, we have

$$|YZ| = \left| BC \cdot \frac{EZ}{EB} \right| = 7 \cdot \frac{\frac{31}{16}}{4} = \frac{217}{64}$$

and

$$|DA| = 2|YZ| = \frac{217}{32}$$

Combinatorics

1. There are 3 singers performing. Each singer has a (possibly empty) set of the other singers that he wishes to perform after. Compute the number of possible values of x such that there are x ways to permute the singers that satisfy their wishes.

Answer: 5

Proposed by Aaryan Vaishya

We can achieve 6 permutations by posing no restrictions on the singers. Observe that any restrictions imposed forces at least half of the permutations to be inaccessible. We can get 0 permutations by imposing contradictory restrictions (first singer wants to go after second but second wants to go after first), 1 permutation from imposing the restriction that the first singer wants to go after everyone else and the second singer wants to go after the third one (this fixes one permutation that works), 2 permutations from imposing the restriction that the first singer wants to go after the second and third singers (the first singer goes at the end and the two other singers can perform in any order), and 3 permutations from imposing the restriction that the first singer wants to go after the second singer (half the permutations should work). Counting our answers, we get $\boxed{5}$

2. Let 10 points be randomly and independently chosen on the circumference of a circle. Compute the expected number of acute triangles formed by the points.

Answer: 30

Proposed by Aaryan Vaishya

Each set of 3 points has a $\frac{1}{4}$ chance of forming an acute triangle (fix point A and randomly choose point B, each measure of arc AB is equally likely, then if we want an acute triangle we must have C between the antipodes of A, B which happens with probability arc AB over 360, averaging this over all equally likely values of arc AB gives $\frac{90}{360} = \frac{1}{4}$), so linearity of expectation gives $\frac{1}{4}\binom{10}{3} = \boxed{30}$.

3. Let f be a function defined as

$$\begin{cases} f(1) = 1 \\ f(x) = \sum_{p|x} f(\frac{x}{p}) \quad x \neq 1, 30 \\ f(30) = 0 \end{cases}$$

Where the sum is taken over all prime divisors p of x. Compute $f(1620000)$.

Answer: 64890

Proposed by Darsh Patel and Xuzhou Ren

Notice that there is a bijection from the value of $f(1620000)$ to the number of paths from $(0,0,0)$ to $(5,4,4)$ not passing through $(1,1,1)$. This is simply $\frac{13!}{5!\cdot 4!\cdot 4!} - 3! \cdot \frac{10!}{4!\cdot 3!\cdot 3!} = \boxed{64890}$.

4. A complete graph on 5 vertices has its edges colored either red or blue randomly. Compute the probability that the graph has exactly one monochromatic triangle.

Answer: $\frac{65}{256}$

Proposed by Aaryan Vaishya

We calculate the total number of colorings with exactly one monochromatic triangle. Fix 3 vertices to form the triangle and the color of the triangle as red, we multiply by 20 to compensate later. For both vertices not in the triangle, we consider the colors to the vertices in the triangle. Note that we cannot have two red edges going to the triangle from a vertex not in the triangle, so there can be at most one red edge for both of them. We can then do casework on how many red edges come out for each vertex not in the triangle (note that the edge between the two vertices not in the triangle is fixed, it must be red since there are 2 blue edges coming out of each of these vertices, so there must be some vertex in the triangle that has a blue edge coming out to each of these vertices). If both have 0 red edges going to the triangle, or one has 0 and the other has 1, no monochromatic triangles are formed, giving a total of $1 + 2 \cdot 3 = 7$ cases. If both have 1 red edge going out, they cannot go out to the same vertex, otherwise we would force the edge between these two vertices to be blue, impossible. This gives a total of $3 \cdot 3 - 3 = 6$, giving a total of 13 cases. There are then $13 \cdot 20 = 260$ desired configurations, dividing by 1024 gives the answer of $\boxed{\frac{65}{256}}$