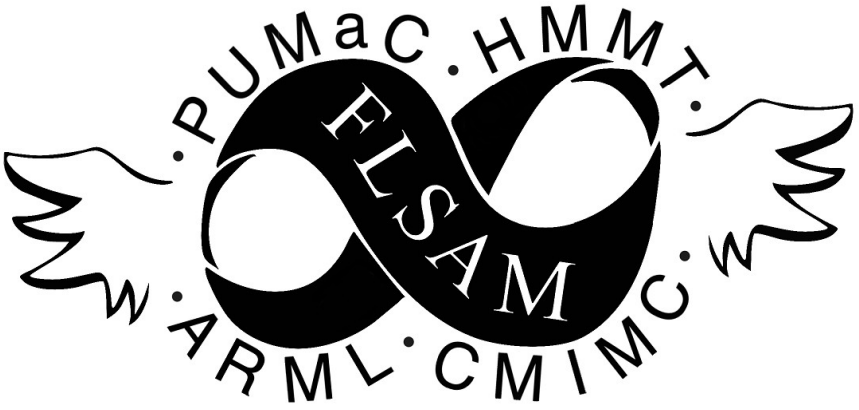
Florida Student Association of Mathematics



2025-2026 Florida Online Math Open

Winter 2025-2026

Solutions

Problem Contributors

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Thank you to everyone who submitted problems!

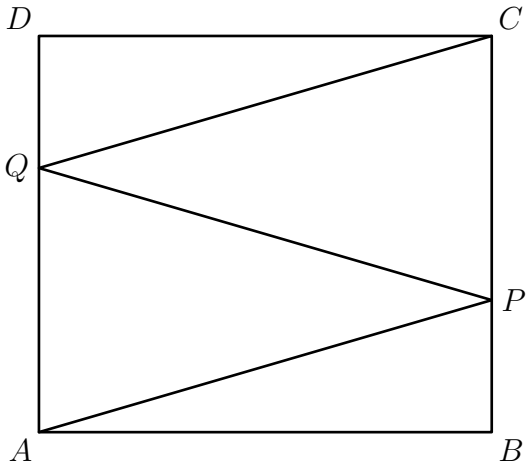
Answers

1. 25
2. 4
3. 1755
4. 50
5. 76820
6. 405
7. 184
8. 23
9. 5859
10. 5280
11. 1122
12. 52
13. 42
14. 115
15. 31250
16. 1159
17. 1440
18. 2193
19. 202502
20. 1048575
21. 1023516
22. 501
23. 1820725
24. 1848015
25. 636
26. 1076
27. 44212
28. 8605
29. 2013844
30. 94

1. Rectangle $ABCD$ has side lengths $AB = 24$ and $BC = 21$. Points P and Q lie on BC and DA, respectively, such that $AP = PQ = QC$. Find PQ.

Proposed by Jacob Khohayting

Answer: 25



Let M and N be the feet of the altitudes from P and Q to AD and BC, respectively. Because $\triangle APQ$ and $\triangle PQC$ are isosceles, then $MA = MQ = NP = NC$. So, $MQ = 7$ and $MP = 24$. By Pythagorean, then $PQ = \boxed{25}$.

2. Bob flips a fair coin until he gets 2 tails in total. Compute the expected number of times he flips the coin.

Proposed by Benjamin Jiang

Answer: 4

Consider the expected number of flips until obtaining a single tail. The probability of flipping a tail is $\frac{1}{2}$, so the expected number of flips is 2. Doubling since we wish for two tails, then the expected number of flips is $\boxed{4}$.

3. Jeffrey writes out the numbers 1 to 2025 from left to right. Between each pair of adjacent digits, Jeffrey either inserts a "$+$" sign or a "$-$" sign. Given that the result of Jeffrey's sum is negative, find the minimum number of "$-$" sign's Jeffrey could have used.

Proposed by Jacob Khohayting

Answer: 1755

We count the number of appearances of each digit. From 1 to 1999, each of the digits 2 to 9 appear 600 times, with 1 appearing an additional 1000 times for 1600 total. We can then check that from 0 to 25 inclusive, each of $(1, 2, \ldots, 9)$ appear $(13, 9, 3, 3, 3, 2, 2, 2, 2)$ times respectively. From 2000 to 2025, 2 also appears 26 times in the 10^3 digit. Finally, the total number of digits from 1 to 2025 is $9 + 2 \cdot 90 + 3 \cdot 900 + 4 \cdot 1026 = 6993$. Computing the final totals for the nonzero digits, summing, and then subtracting from this total yields the final total for 0. This gives

$$(528, 1613, 635, 603, 603, 603, 602, 602, 602, 602)$$

appearances for $(0, 1, 2, \ldots, 9)$ respectively. The total digital sum if Jeffrey uses all $+$'s is

$$0 \cdot 528 + 1 \cdot 1613 + 2 \cdot 635 + 3 \cdot 603 + 4 \cdot 603 + 5 \cdot 603 + 6 \cdot 602 + 7 \cdot 602 + 8 \cdot 602 + 9 \cdot 602 = 28179$$

Now placing $-$ signs decreases this total by twice the amount Jeffrey is subtracting in his expression. Clearly, his optimal strategy is to subtract the largest digits first. We can check that

$$28179 - 2 \cdot (9 \cdot 602 + 8 \cdot 602 + 7 \cdot 551) = -3$$

gives the optimal placement of $-$'s, for a total of $602 + 602 + 551 = \boxed{1755}$.

4. Find the number of six-digit positive integers N which satisfy the following properties:

- N is divisible by 2 and exactly two of its digits are 2; and
- N is divisible by 3 and exactly three of its digits are 3.

Proposed by Jacob Khohayting

Answer: 50

Note that N has digits 2, 2, 3, 3, 3, and K. Since N is divisible by 3, then $K \equiv 2 \pmod 3$. So, $K \in \{2, 5, 8\}$. But, $K \neq 2$ otherwise we would contradict the first condition. We have two cases:

- If $K = 5$, then the last digit is 2. The five other digits can be 2, 3, 3, 3, 3, or 5. There are $\frac{5!}{3!1!1!}$ ways for this case.
- If $K = 8$, then we have two other cases:
  - If the last digit is 2, then the other five digits are 2, 3, 3, 3, and 5 yielding $\frac{5!}{3!1!1!}$ ways for this case.
  - If the last digit is 8, then the other five digits are 2, 2, 3, 3, and 3 yielding $\frac{5!}{3!2!}$ ways for this case.

Hence, the total number of ways is:

$$\frac{5!}{3!1!1!} + \frac{5!}{3!1!1!} + \frac{5!}{3!2!} = 20 + 20 + 10 = \boxed{50}.$$

5. A real number α is chosen randomly and uniformly from the interval $[2, 5]$. Let p be the probability that 17 appears in the following sequence:

$$\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \lfloor 4\alpha \rfloor, \ldots$$

If p can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Roger He

Answer: 76820

The integer 17 appears in the sequence if and only if there exists a positive integer n such that

$$17 \leq n\alpha < 18 \iff \frac{17}{n} \leq \alpha < \frac{18}{n}.$$

Thus, for each n, the values of α that cause $\lfloor n\alpha \rfloor = 17$ lie in the interval

$$\left(\frac{17}{n}, \frac{18}{n} \right).$$

We now determine which values of n give intervals that intersect the interval $(2, 5)$.

- For the interval to reach above 2, we need $\frac{18}{n} > 2$, which gives $n < 9$.
- For the interval to reach below 5, we need $\frac{17}{n} < 5$, which gives $n > \frac{17}{5}$.

Therefore the possible integers are $n = 4, 5, 6, 7, 8$. For these values, the corresponding intervals are

$$\left(\frac{17}{4}, \frac{18}{4}\right), \left(\frac{17}{5}, \frac{18}{5}\right), \left(\frac{17}{6}, \frac{18}{6}\right), \left(\frac{17}{7}, \frac{18}{7}\right), \left(\frac{17}{8}, \frac{18}{8}\right).$$

Each interval lies inside $(2, 5)$, and they do not overlap. The total length of all values of α for which 17 appears is therefore

$$\sum_{n=4}^{8} \left(\frac{18}{n} - \frac{17}{n}\right) = \sum_{n=4}^{8} \frac{1}{n} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$$

Using a common denominator,

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{210 + 168 + 140 + 120 + 105}{840} = \frac{743}{840}.$$

Since α is chosen uniformly from an interval of total length $5 - 2 = 3$, the desired probability is $\frac{1}{3} \cdot \frac{743}{840} = \frac{743}{2520}$. Hence, the answer is $\boxed{76820}$.

6. Darsh plays a game starting with the number 0 and a set $S = \{1, 2, 4, \ldots, 512, 1024\}$. Every second, he selects an element of S at random, removes it from S, and adds it to his number. Let p be the probability that his new number is greater than 100 after 3 seconds. If p can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Jacob Khohayting

Answer: 405

Note that $100 = 00001100100_2$. There are two ways to get a number greater than 100:

- Either at least one number greater than or equal to 128. To calculate this, we compute the complement. The probability all numbers are less than or equal to 64 is

$$P(\text{All numbers } \leq 64) = \frac{7 \cdot 6 \cdot 5}{11 \cdot 10 \cdot 9} = \frac{7}{33}.$$

  Thus, the probability at least one of the numbers is at least 128 is $1 - \frac{7}{33} = \frac{26}{33}$.
- Or, we get a 64, a 32, and either 16 or 8. Here, there are 2 ways to choose whether we have 16 or 8. Then, there are 3! ways to permutate our chances. Thus, the probability our number is greater than 100 is

$$\frac{2 \cdot 3!}{11 \cdot 10 \cdot 9} = \frac{2}{165}.$$

Summing, then $p = \frac{26}{33} + \frac{2}{165} = \frac{4}{5}$. Thus, the answer is $\boxed{405}$.

7. Jason's favorite number is

$$\underbrace{12121212\ldots12121212}_{\text{2026 digits}}.$$

He inserts a 0 between some two consecutive digits such that the new number is divisible by 11. Over all 2025 possible insertions, find the number of possible ways he could have inserted a 0.

Proposed by Jacob Khohayting

Answer: 184

Let N be the large number. Currently, $N \equiv 1 \pmod{11}$.

- Suppose we insert the 0 after a 2. Suppose the digits after the 0 form some number M, which is divisible by 11. Subtracting the new number and the original number, then:

$$10 \cdot M - M \equiv -1 \pmod{11}$$

Thus, $M \equiv 6 \pmod{11}$. Notice, M is of the form $1212\cdots12$. If M has $2k$ digits, then

$$M \equiv -1 + 2 - 1 + 2 - \cdots - 1 + 2 \equiv k \pmod{11}.$$

Thus, $k \equiv M \equiv 6 \pmod{11}$. So, $2k \equiv 12 \pmod{22}$. Thus, the number of digits in M is $12 \pmod{22}$. Hence, all $12 + 22k$ for $0 \leq k \leq 91$ work yielding 92 ways.

- Suppose we insert the 0 after a 1. Suppose the digits after the 0 form some number M, which is divisible by 11. Then,

$$10 \cdot M - M \equiv -1 \pmod{11}$$

Thus, $M \equiv 6 \pmod{11}$. Notice, M is of the form $1212\cdots121$. If M has $2k+1$ digits, then

$$M \equiv 1 - 2 + 1 - 2 + \cdots + 1 - 2 + 1 \equiv 1 - k \pmod{11}.$$

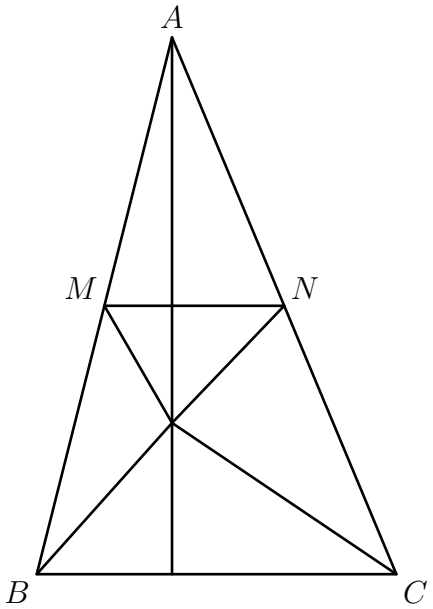
Thus, $1 - k \equiv M \equiv 6 \pmod{11}$ thus $k \equiv 6 \pmod{11}$. So, $2k + 1 \equiv 13 \pmod{22}$. Thus, the number of digits in M is $13 \pmod{22}$. Hence, all $13 + 22k$ for $0 \leq k \leq 91$ work yielding 92 options.

Together, we have $92 + 92 = \boxed{184}$ total options.

8. Let ABC be a triangle with orthocenter H such that M and N are midpoints of AB and AC, respectively. Suppose that $MH = 4$, $BH = 6$, and $CH = 8$. Find HN^2.

Proposed by Jacob Khohayting

Answer: 23



Let the foot of the altitude from A to MN and BC by P and Q, respectively. Then:

$$BH^2 - BQ^2 = CH^2 - CQ^2 \iff CQ^2 - BQ^2 = CH^2 - BH^2 = 28.$$

But, MN is the midline, so $NP^2 = MP^2 = 7$. Then,

$$HN^2 - NP^2 = HM^2 - MP^2 \iff HN^2 = HM^2 + NP^2 - MP^2 = 4^2 + 7 = \boxed{23}.$$

9. Arnav's favorite three-digit number is $\overline{ABC}$, where $A \neq 0$. When he interprets this as a base b number, he finds that it is divisible by 31_b for $b = 10, 13, 16$. Compute the sum of all distinct possible values of $\overline{ABC}$ (in base 10).

Proposed by Jacob Khohayting

Answer: 5859

The condition implies

$$\begin{aligned}(a_2a_1a_0)_b &= a_2 \cdot b^2 + a_1 \cdot b^1 + a_0 \cdot b^0 \\ &\equiv a_2 \cdot \left(-\tfrac{1}{3}\right)^2 + a_1 \cdot \left(-\tfrac{1}{3}\right)^1 + a_0 \cdot \left(-\tfrac{1}{3}\right) \pmod{3b+1} \\ &\equiv \frac{a_2 - 3a_1 + 9a_0}{9} \pmod{3b+1}.\end{aligned}$$

Therefore, we require that

$$\frac{a_2 - 3a_1 + 9a_0}{9}$$

to be divisible by the least common multiple of 31, 40, and 49. Essentially, $a_2 - 3a_1 + 9a_0 = 0$. Note that $3 \mid a_2$, so we consider the following cases:

- If $a_2 = 3$, then $1 - a_1 + 3a_0 = 0$, so $a_1 = 3a_0 + 1$ yielding the possible values of 310, 341, and 372.
- If $a_2 = 6$, then $2 - a_1 + 3a_0 = 0$, so $a_1 = 3a_0 + 2$ yielding the possible values of 620, 651, and 682.
- If $a_2 = 9$, then $3 - a_1 + 3a_0 = 0$, so $a_1 = 3a_0 + 3$ yielding the possible values of 930, 961, and 992.

Therefore, the sum of all possible values is $651 \cdot 9 = \boxed{5859}$.

10. George has an infinite supply of indistinguishable white pieces and indistinguishable black pieces. Find the number of ways to place 10 pieces on a 5×5 grid so that

- each square contains at most one piece;
- no two white pieces are in the same row or column; and
- no two black pieces are in the same row or column.

Proposed by inferno2332

Answer: 5280

The conditions imply that we wish to place 5 white and 5 black pieces.

There are $5! = 120$ valid ways to place the white pieces as for each permutation $(a_1, a_2, \cdots, a_5)$ of $(1, 2, 3, 4, 5)$, the white piece in the nth row will go in the a_nth column.

There are $!5 = 44$ valid ways to place the black pieces as for each derangement $(b_1, b_2, \cdots, b_5)$ of $(a_1, a_2, \cdots, a_5)$, the black piece in the nth row will go in the b_nth column.

The total is thus $120 \cdot 44 = \boxed{5280}$.

11. Call a positive integer *n brainy* if it cannot be written in the form $67a + 68b + 69c$ for nonnegative integers a, b, and c. Compute the number of brainy positive integers.

Proposed by Roger He

Answer: 1122

Consider the set of values attained by $67a + 68b + 69c$ for a fixed value of $n = a + b + c$. Clearly all such values lie in $[67n, 69n]$. We can also start with $(a, b, c) = (n, 0, 0)$ and repeatedly add $(-1, 1)$ to either (a, b) or (b, c) to increase the value of $67a + 68b + 69c$ by 1 until it reaches $69n$. Then it achieves all integer values in $[67n, 69n]$.

Then the brainy numbers are the positive integers not in one of the intervals $[67n, 69n]$ for $n \geq 0$. The number of positive integers between $[67n, 69n]$ and $[67(n+1), 69(n+1)]$ is $66 - 2n$ as long as this is nonnegative. The number of brainy numbers is then

$$\sum_{\substack{n \geq 0 \\ 66-2n \geq 0}} (66 - 2n) = \sum_{n=0}^{33} 2n = 33 \cdot 34 = \boxed{1122}$$

12. Find the number of 5-tuples (F, L, S, A, M) of digits such that $F \cdot L$, $L \cdot S$, $S \cdot A$, and $A \cdot M$ all have units digit 6.

Proposed by Jacob Khohayting

Answer: 52

Notice, we have three components: $\{1, 6\}$, $\{4, 9\}$, and $\{2, 3, 7, 8\}$. We consider each of these components separately:

Firstly, consider the component $1 \implies 6$ and $6 \implies \{1, 6\}$. This implies no two consecutive values are 1. The number of ways is simply $F_{5+2} = 13$ total ways.

Now, consider the component $9 \implies 4$ and $4 \implies \{4, 9\}$. This is equivalent to above yielding 13 total ways.

Finally, consider the component $3 \iff 2 \iff 8 \iff 7$. Let X_n be the number of sequences of length n ending in 2. Note this is equivalent to the number of sequences ending in 8. Let Y_n be the number of sequences of length n ending in 3. Note this is equivalent to the number of sequences ending in 7. Now, we will derive recursive equations:

- If the nth term of the sequence is 3, then the $(n-1)$th term is 2. Therefore, $Y_{n+1} = X_n$.
- If the nth term of the sequence is 2, then the $(n-1)$th term is 3 or 8. Therefore, $X_{n+1} = Y_n + X_n$.

Thus, $Y_{n+1} = Y_n + Y_{n-1}$. We can derive that $Y_0 = 0$ and $Y_1 = 1$. Thus, $\{Y\}$ is the Fibonacci sequence, whereas $\{X\}$ is a shifted Fibonacci sequence. Thus, the number of sequences is

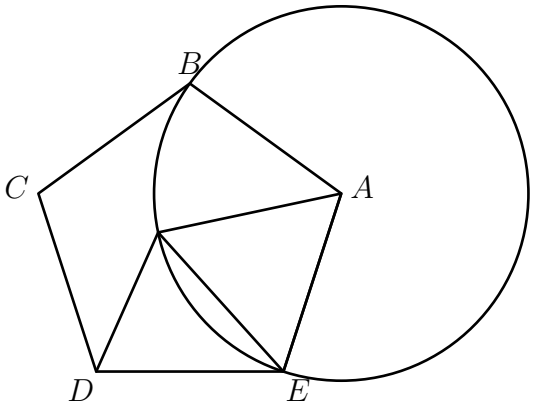
$$2X_5 + 2Y_5 = 2F_6 + 2F_5 = 2F_7 = 2 \cdot 13 = 26.$$

Thus, the total number of sequences is $13 + 13 + 26 = \boxed{52}$. In general, the answer is $4 \cdot F_{n+2}$ where F_n is the nth term of the Fibonacci sequence.

13. Let P be a point inside regular pentagon $ABCDE$ such that $\angle APE = 60^\circ$ and $\angle BPA = 66^\circ$. Find the degree measure of $\angle PDC$.

Proposed by George Paret

Answer: 42



Notice,

$$\angle EPB + \frac{\angle BAE}{2} = 126^\circ + \frac{108^\circ}{2} = 180^\circ.$$

Hence, P lies on (ABE).

As $\angle APE = 60^\circ$, then $DE = AE = EP$. Thus, $\triangle DEP$ is isosceles. Angle chasing:

$$\begin{aligned} &\angle PED = \angle AED - \angle AEP = 48^\circ \\ \implies &\angle PDE = 66^\circ \\ \implies &\angle PDC = 108^\circ - 66^\circ = \boxed{42}^\circ. \end{aligned}$$

14. Let a, b, and c be non-zero real numbers such that

$$\frac{a^2c + c^2b + b^2a}{7} = \frac{a^2b + b^2c + c^2a}{11} = abc.$$

Compute $\dfrac{a^3}{b^3} + \dfrac{b^3}{c^3} + \dfrac{c^3}{a^3}$.

Proposed by Joshua Khohayting

Answer: 115

Substitute $(x, y, z) = (\frac{a}{b}, \frac{b}{c}, \frac{c}{a})$. The conditions imply

$$\begin{aligned} x + y + z &= 7, \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 11, \\ xyz &= 1. \end{aligned}$$

Then,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 11 \iff \frac{xy + yz + zx}{xyz} = 11 \iff xy + yz + zx = 11.$$

Therefore,

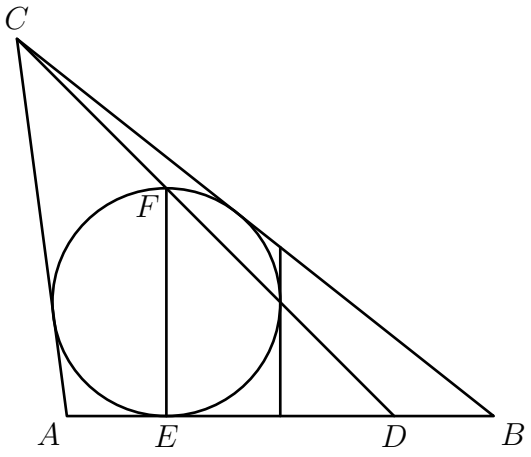
$$\begin{aligned} x^3 + y^3 + z^3 &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz \\ &= (x + y + z)\big((x + y + z)^2 - 3(xy + yz + zx)\big) + 3xyz \\ &= 7 \cdot (7^2 - 3 \cdot 11) + 3 \\ &= \boxed{115}. \end{aligned}$$

15. Let ABC be a triangle with incircle ω. The perpendicular bisector of AB is tangent to ω and an excircle is tangent to segment AB at D. Given that $AB = 20$ and $CD = 25$, find the square of the area of ABC.

(An *excircle* is the circle tangent to one side of the triangle and the extensions of the other two sides.)

Proposed by Hanfei He

Answer: 31250



Let E be the point of tangency between the incircle and AB. Let F be the point diametrically opposite E. Finally, denote T as the intersection of the perpendicular bisector of AB and ω. Denote r as the inradius of ω.

Length chasing, then $IF = IE = IT = r$. So, if M is the midpoint of AB, then $EM = DM = r$. But, $\angle FED = 90^\circ$ and $FE = ED$, so $\angle FDE = 45^\circ$.

Henceforth, $[ABC] = \frac{1}{2} \cdot AB \cdot CD \cdot \angle CDE = 125\sqrt{2}$. Thus, the desired answer is $\boxed{31250}$.

16. Primes p and q are such that $\frac{3q+4}{p}$ and $\frac{4p+3}{q}$ are integers. Compute the sum of all distinct possible values of pq.

Proposed by Jacob Khohayting

Answer: 1159

First, suppose $p \geq q$. If $p = 2$, then $q = 2$. But, then $\frac{4p+3}{q}$ is not an integer. If $p = 3$, then $\frac{3q+4}{p}$ is not an integer. Suppose $p \geq 5$, then:

$$\frac{3q+4}{p} \leq \frac{3p+4}{p} = 3 + \frac{4}{p} < 4.$$

We consider the following cases:

- If $\frac{3q+4}{p} = 3$, then $3q + 4 = 3p$, so $3(p - q) = 4$. Clearly, this is impossible.
- If $\frac{3q+4}{p} = 2$, then $3q + 4 = 2p$, so $3q = 2(p - 2)$. Thus, $q = 2$. But, then $\frac{4p+3}{q}$ is not an integer, so this is impossible.
- If $\frac{3q+4}{p} = 1$, then $3q + 4 = p$, so
$$\frac{4p+3}{q} = \frac{12q+19}{q} = 12 + \frac{19}{q}.$$
Hence, $q = 19$. Then, $p = 61$. Clearly, this works.

Now, suppose $q \geq p$. If $q = 2$, there are no solutions since $\frac{4p+3}{q}$ is not an integer. If $q = 3$, then $p = 3$. But, then $\frac{3q+4}{q}$ is not an integer. Suppose $q \geq 5$, then:

$$\frac{4p+3}{q} \leq \frac{4q+3}{q} = 4 + \frac{3}{q} < 5.$$

Now, consider the following cases:

- If $\frac{4p+3}{q} = 4$, then $4(q-p) = 3$, impossible.
- If $\frac{4p+3}{q} = 3$, then $3q - 3 = 4p$. Thus, $p = 3$, but this is impossible as then $\frac{3q+4}{p}$ is not an integer.
- If $\frac{4p+3}{q} = 2$, then $4p + 3 = 2q$. Impossible as this requires $2 \mid 3$.
- If $\frac{4p+3}{q} = 1$, then $4p + 3 = q$. Then,

$$\frac{3q+4}{p} = \frac{12p+13}{p} = 12 + \frac{13}{p}.$$

  Thus, $p = 13$. Substituting, then $q = 55$, but this isn't prime. Therefore, there are no solutions for this case.

Hence, we have shown there exists a unique solution $(61, 19)$. Thus, $pq = \boxed{1159}$.

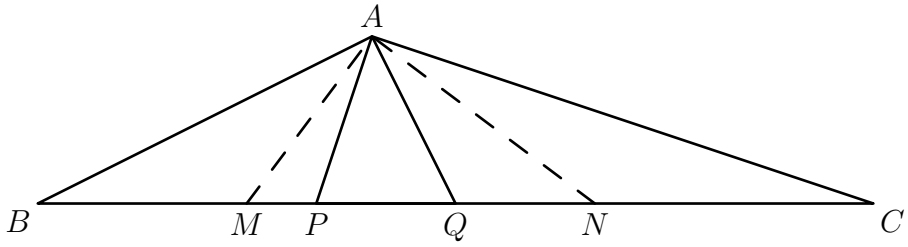
17. Let ABC be a triangle with points P and Q on segment BC with B, P, Q, C occurring in that order such that

$$\angle BAP = \angle PAQ = \angle QAC = 45^\circ.$$

Given that $BP = 40$ and $QC = 60$, compute the area of ABC.

Proposed by Jacob Khohayting

Answer: 1440



Let M and N be midpoints of BQ and PC, respectively. Then, $\angle MAN = 90^\circ$. Let $PQ = 20A$. So:

$$MQ = \frac{BQ}{2} = \frac{BP + PQ}{2} = \frac{40 + 20A}{2} = 20 + 10A,$$
$$NP = \frac{CP}{2} = \frac{CQ + QP}{2} = \frac{60 + 20A}{2} = 30 + 10A.$$

Then, by Pythagorean, we find:

$$MA^2 + AN^2 = MN^2$$
$$MQ^2 + NP^2 = MN^2$$
$$(20 + 10A)^2 + (30 + 10A)^2 = (50)^2$$
$$(2 + A)^2 + (3 + A)^2 = 5^2$$

Hence, $A = 1$. Thus, $\triangle MAN$ has side lengths 30, 40, and 50. Hence, $d(A, MN) = \frac{30 \cdot 40}{50} = 24$. Therefore, we can compute:

$$[ABC] = \frac{d(A, MN) \cdot BC}{2} = \frac{d(A, MN) \cdot (BP + PQ + QC)}{2} = \frac{24 \cdot (40 + 20 + 60)}{2} = \boxed{1440}.$$

18. For any positive integer n, let $f_3(n)$ be the largest power of 3 that divides n. For example, $f_3(7) = 1$ and $f_3(54) = 27$. Let $S = \sum_{n=1}^{\infty} \frac{f_3(n)}{n^2}$. Compute $\lfloor 1000S \rfloor$.

Proposed by inferno2332

Answer: 2193

The sum can be decomposed as

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \left(\frac{2}{3^2} + \frac{2}{6^2} + \frac{2}{9^2} + \cdots\right) + \left(\frac{6}{9^2} + \frac{6}{18^2} + \frac{6}{27^2} + \cdots\right) + \cdots$$

This is equal to

$$\left(1 + \frac{2}{9} + \frac{6}{27} + \frac{18}{729} + \cdots\right)\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = \frac{4}{3} \cdot \frac{\pi^2}{6} = \frac{2\pi^2}{9}.$$

So the answer is $\lfloor 1000 \cdot \frac{2\pi^2}{9} \rfloor = \boxed{2193}$.

19. Roger chooses a random permutation $(S_1, S_2, \ldots, S_{2^{2025}})$ of the subsets of $\{1, 2, \ldots, 2025\}$. He now takes $A_1 = S_1$, and for $1 < k \leq 2^{2025}$ randomly chooses A_k to be either $A_{k-1} \cup S_k$ or $A_{k-1} \cap S_k$ with $\frac{1}{2}$ probability of either choice. Let E be the expected number of elements of $A_{2^{2025}}$. If E can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Roger He

Answer: 202502

Let X denote the space of all orderings of S_n and choices of intersection or union. Notice that by De Morgan's laws, there is a symmetry in the set; if you swap unions with intersections and sets with their complements, you get a bijection on X which preserves the average of $|A_{2^{2025}}|$. But also note that if $\mathbb{E}(|A_{2_{2025}|}) = a$, then this bijection would take a to $2025 - a$. Thus, $a = \frac{2025}{2}$. Hence, the answer is $\boxed{202502}$.

20. There are 20 coins laid out on a table, with one of them showing tails and the rest showing heads. Every second, a coin is chosen at random and flipped over to reveal the other side. Find the expected number of seconds until all the coins show heads.

Proposed by Hanfei He

Answer: 1048575

Let $N = 20$. We use states. Let E_k be the expected amount of seconds for the process to get from a starting state with k coins heads up to one with $k + 1$ coins heads up. We wish to compute E_{N-1}. In our starting state, we get to $k + 1$ heads in 1 second with probability $\frac{N-k}{N}$. Otherwise, we get to $k - 1$ heads, and then wait an average of E_{k-1} seconds to get back to k heads. This yields

$$E_k = 1 + \left(\frac{k}{N}\right)(E_{k-1} + E_k) \implies E_k = \frac{k}{N-k}E_{k-1} + \frac{N}{N-k} \implies \frac{E_k + 1}{2} = \frac{k}{N-k}\left(\frac{E_{k-1} + 1}{2}\right) + 1$$

Now, we have $E_0 = 1$. Our recursive formula yields

$$\begin{aligned}
\frac{E_0+1}{2} &= 1 \\
\frac{E_1+1}{2} &= \frac{1}{N-1}+1 \\
\frac{E_2+1}{2} &= \left(\frac{1}{N-1}\right)\left(\frac{2}{N-2}\right)+\frac{2}{N-2}+1 \\
\frac{E_3+1}{2} &= \left(\frac{1}{N-1}\right)\left(\frac{2}{N-2}\right)\left(\frac{3}{N-3}\right)+\left(\frac{2}{N-2}\right)\left(\frac{3}{N-3}\right)+\frac{3}{N-3}+1 \\
&\vdots \\
\frac{E_{N-1}+1}{2} &= \left(\frac{1}{N-1}\right)\cdots\left(\frac{N-1}{1}\right)+\left(\frac{2}{N-2}\right)\cdots\left(\frac{N-1}{1}\right)+\left(\frac{3}{N-3}\right)\cdots\left(\frac{N-1}{1}\right)+\cdots+1
\end{aligned}$$

Our expression for $\frac{E_{N-1}+1}{2}$ is just $\binom{N-1}{N-1}+\binom{N-1}{N-2}+\binom{N-1}{N-3}+\cdots+\binom{N-1}{0}$, which is 2^{N-1}. Then $E_{N-1} = 2^N - 1 = \boxed{1048575}$.

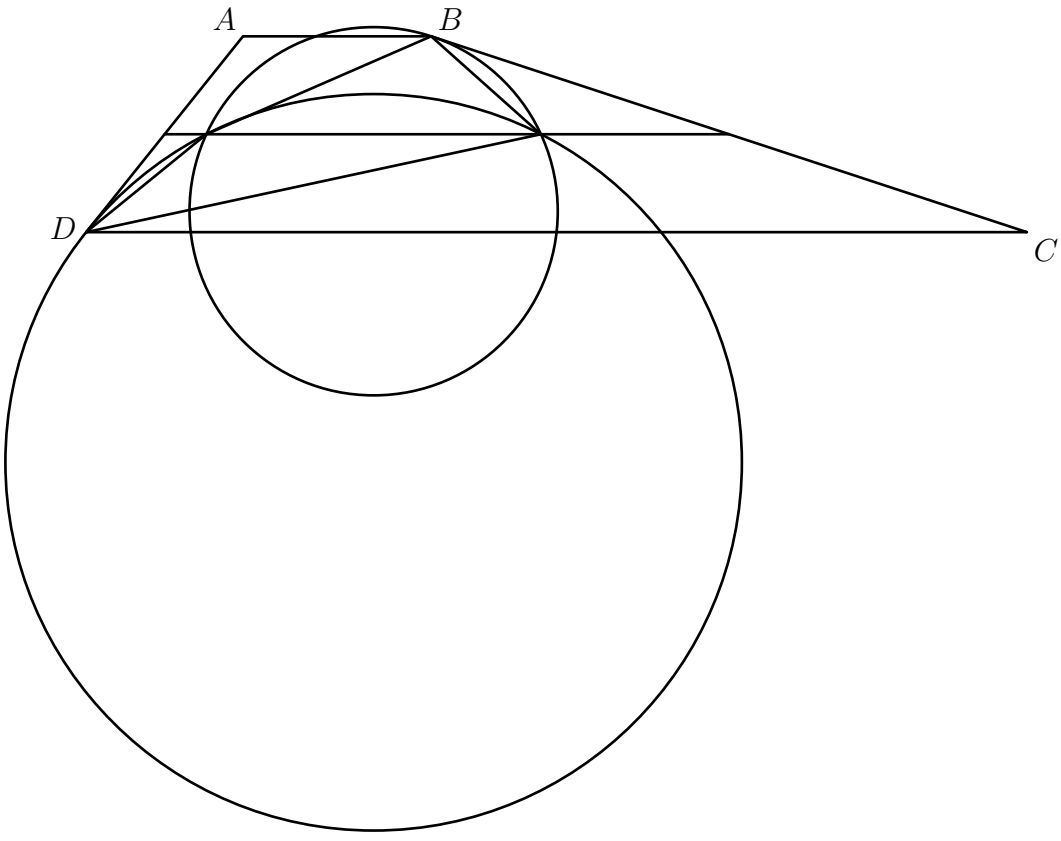
21. Let $ABCD$ be a trapezoid with bases AB and CD having side lengths $AB = 18$ and $CD = 90$. Let M and N be the midpoints of AD and BC, respectively. Points P and Q lying on MN with M, P, Q, N appearing in that order are such that

$$\angle ABQ = \angle PBC \quad \text{and} \quad \angle CDP = \angle QDA.$$

Given that $MP = 4$ and $QN = 18$, compute the square of the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: 1023516



First, $\angle ABQ = \angle PBC$ implies $\angle ABP = \angle NBQ$. So,

$$\angle NBQ = \angle ABQ = \angle BPN.$$

Thus, (BPQ) is tangent to BN. Likewise, (DPQ) is tangent to DM.

Since MN is the midline, then $MN = \frac{AB+CD}{2} = 54$. Hence, $PQ = MN - MP - QN = 32$. By Power of Point, then

$$BN^2 = NQ \cdot NP = 18 \cdot 50 \implies BN = 30,$$
$$DM^2 = MP \cdot MQ = 4 \cdot 36 \implies DM = 12.$$

Hence, $AB = 18$, $BC = 60$, $CD = 90$, and $DA = 24$.

Let the feet of the perpendiculars from A and B to CD be P and Q, respectively. Denote the height as h and $QC = x$, then:

$$60^2 = BC^2 = BQ^2 + QC^2 = h^2 + x^2,$$
$$24^2 = DA^2 = AP^2 + PD^2 = h^2 + (72 - x)^2.$$

Solving, then $h = 3\sqrt{39}$. Thus, the area is $[ABCD] = \frac{AB+CD}{2} \cdot h = 162\sqrt{39}$. So, the answer is $\boxed{1023516}$.

22. A positive integer n is called *lucid* if $n > 1$ and there do not exist positive integers $a, b, c, d < n$ for which

$$a^2b^4c^3d^5 \gcd(a^2 + b^4, c^3 + d^5) \cdot 2^a 4^b 3^c 5^d \operatorname{lcm}(2^a + 4^b, 3^c + 5^d)$$

is divisible by n. Compute the sum of all lucid n less than 300.

Proposed by Darsh Patel

Answer: 501

We first claim that all lucid n are $3 \pmod 4$ primes. If n were composite, then we could set $ab = n$ to get $n \mid a^2b^4c^3d^5$, and so n is prime. We have that n is odd since otherwise we could set $a = 1$. If $n \equiv 1 \pmod 4$, then since -1 is a quadratic residue modulo all $1 \pmod 4$ primes we could force $a^2 \equiv -1 \pmod n$, set $b = 1$, $c = 1$, and $d = n - 1$ to get $n \mid \gcd(a^2 + b^4, c^3 + d^5)$, and so $n \equiv 3 \pmod 4$.

Then we always have $n \nmid a^2b^4c^3d^5$. Also, since $\left(\frac{-1}{n}\right) = -1$, a sum of two quadratic residues cannot be 0 modulo n. Then $n \nmid a^2 + b^4$, and so $n \nmid \gcd(a^2 + b^4, c^3 + d^5)$. We now also note that $n \neq 3$, since otherwise we could set $c = 1$. The condition now reduces to n not dividing $2^a + 4^b$ or $3^c + 5^d$ for any $a, b, c, d < n$. We can treat a, b, c, d as residues modulo $n - 1$.

Setting $b \equiv 0 \pmod{n-1}$, we get $2^a \not\equiv -1 \pmod n$ for any a. This implies that 2 is a quadratic residue modulo n, since otherwise $a = \frac{n-1}{2}$ would work. A similar argument yields that 4 is a quadratic residue. This is also sufficient to imply that $2^a + 4^b \not\equiv 0 \pmod n$ always. A similar argument shows that the condition for $n \nmid 3^c + 5^d$ is equivalent to $\left(\frac{3}{n}\right) = \left(\frac{5}{n}\right) = 1$.

Now clearly $\left(\frac{4}{n}\right) = 1$ always. We have $\left(\frac{2}{n}\right) = 1$ for $1, 7 \pmod 8$ primes, and since $n \equiv 3 \pmod 4$, we need $n \equiv 7 \pmod 8$. By Quadratic Reciprocity, we have

$$\left(\frac{3}{n}\right) = 1 \implies \left(\frac{n}{3}\right) = -1 \implies n \equiv 2 \pmod 3$$
$$\left(\frac{5}{n}\right) = 1 \implies \left(\frac{n}{5}\right) = 1 \implies n \equiv 1, 4 \pmod 5$$

We then get $n \equiv 71, 119 \pmod{120}$. The sum of the primes less than 300 satisfying this is

$$71 + 191 + 239 = \boxed{501}$$

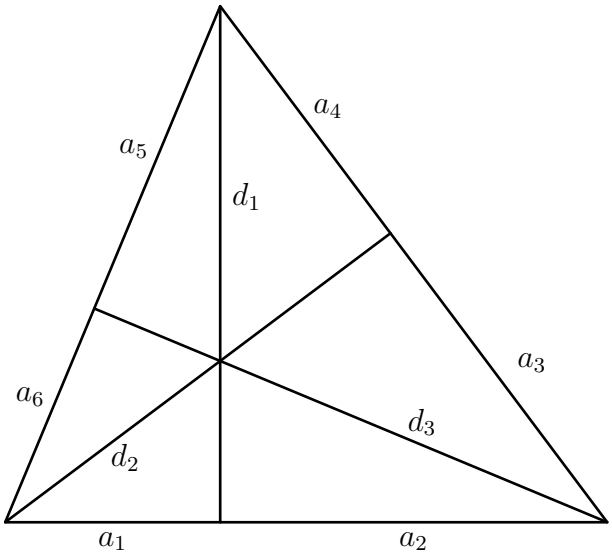
23. Positive real numbers $a_1, a_2, a_3, a_4, a_5, a_6$ and d_1, d_2, d_3 are such that

$$(a_1 + a_2)^2 = a_3^2 + d_2^2 = a_6^2 + d_3^2 = 289$$
$$(a_3 + a_4)^2 = a_5^2 + d_3^2 = a_2^2 + d_1^2 = 625$$
$$(a_5 + a_6)^2 = a_1^2 + d_1^2 = a_4^2 + d_2^2 = 676.$$

If $d_1 + d_2 + d_3$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Luis Fonseca

Answer: 1820725



Consider the above geometric interpretation. Note that d_1, d_2, d_3 are simply the heights of the triangle with sides $s_1 = 17$, $s_2 = 25$, and $s_3 = 26$.

By Heron's, the area is

$$K = \sqrt{\frac{17+25+26}{2} \cdot \frac{-17+25+26}{2} \cdot \frac{17-25+26}{2} \cdot \frac{17+25-26}{2}} = 204.$$

Thus,

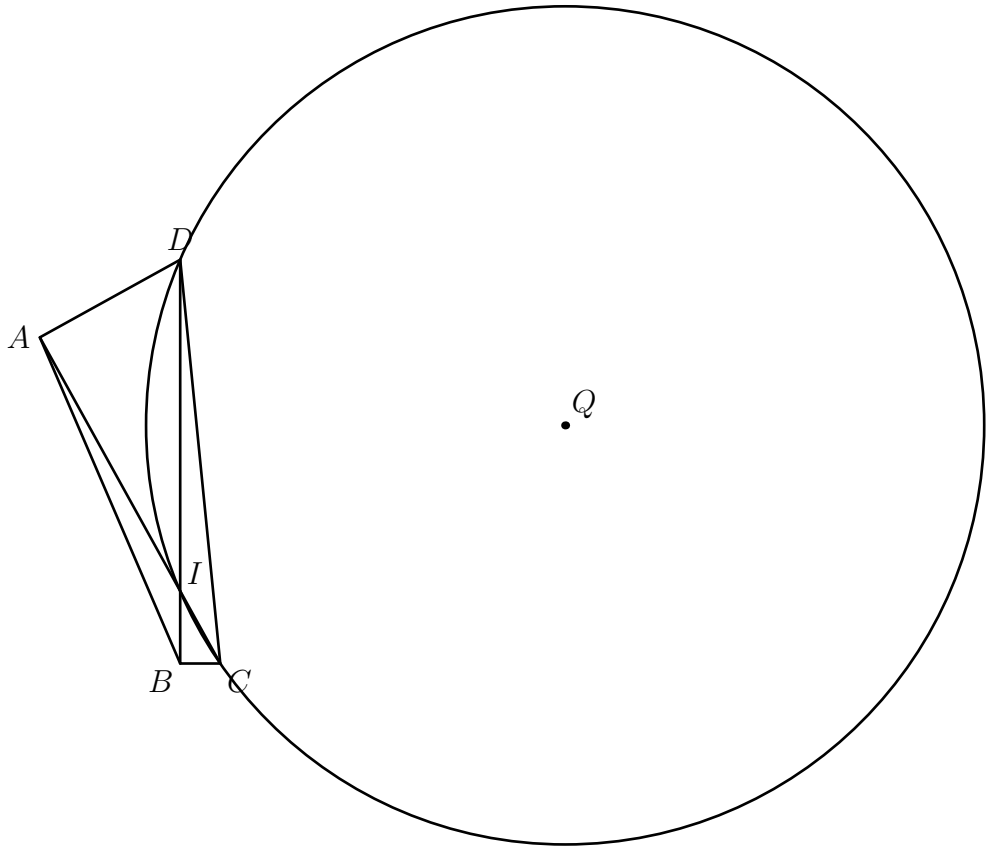
$$d_1 + d_2 + d_3 = \frac{2K}{s_1} + \frac{2K}{s_2} + \frac{2K}{s_3} = 408 \cdot \left(\frac{1}{17} + \frac{1}{25} + \frac{1}{26}\right) = \frac{18204}{325}.$$

Hence, the answer is $\boxed{1820725}$.

24. Convex quadrilateral $ABCD$ has diagonals $AC = 72$ and $BD = 78$ which intersect at I. Let P, Q, and R be the circumcenters of triangles BIC, CID, and DIA respectively. It is given that $QI \perp AB$, $PI = 8$, and $RI = 32$. Compute the square of the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: 1848015



Let M be the midpoint of IC and N be the foot of the perpendicular from QI to AB. Note that

$$\begin{aligned}\angle DIC &= \tfrac{1}{2}\angle IQC \\ &= \angle IQM \\ &= 90 - \angle QIC \\ &= 90 - \angle AIN \\ &= \angle IAB.\end{aligned}$$

Thus, $ABCD$ is cyclic.

Note that $\triangle BIC \sim \triangle AID$. We can write $BI = a$, $CI = b$, $AI = 4a$, and $DI = 4b$. As $AC = 72$ and $BD = 78$, then $a = 14$ and $b = 16$. Note the circumradius of $\triangle BIC$ is 8, so $\angle IBC = \angle IAD = 90^\circ$. Thus, $\sin(\angle BIC) = \frac{2\sqrt{55}}{16}$. Hence, the area is

$$[ABCD] = \tfrac{1}{2} \cdot AC \cdot BD \cdot \sin(\angle BIC) = \tfrac{1}{2} \cdot 72 \cdot 78 \cdot \tfrac{2\sqrt{55}}{16} = 351\sqrt{15}.$$

Therefore, the answer is $\boxed{1848015}$.

25. Define a sequence $\{a_n\}_{n\geq 1}$ of reals by $a_1 = 2$ and

$$a_{n+1} = 2 - \sqrt{4 - a_n}$$

for $n \geq 1$. Let $P = \displaystyle\prod_{n=1}^{\infty} \frac{\sqrt{4-a_n}}{2}$. Compute $\lfloor 1000P \rfloor$.

Proposed by inferno2332

Answer: 636

Let $a_n = 4\sin^2\theta_n$. Then,

$$\begin{aligned} a_{n+1} &= 2 - \sqrt{4 - 4\sin^2\theta_n} \\ 4\sin^2(\theta_{n+1}) &= 2 - 2\cos\theta_n \\ \sin(\theta_{n+1}) &= \sqrt{\frac{1-\cos\theta_n}{2}} = \sin\left(\frac{\theta_n}{2}\right), \end{aligned}$$

by the half-angle formula.

We know $a_1 = 2$ so $\theta_1 = \frac{\pi}{4}$. So $a_n = 4\sin^2\left(\frac{\pi}{2^{n+1}}\right)$ and $\sqrt{4-a_n} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$. Therefore,

$$\begin{aligned} \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots &= \lim_{n\to\infty}\prod_{k=1}^{n}\cos\left(\frac{\pi}{2^{k+1}}\right) \\ &= \lim_{n\to\infty}\frac{2^{-n}\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2^{n+1}}\right)} \\ &= \lim_{n\to\infty}\frac{1}{2^n\sin\left(\frac{\pi}{2^{n+1}}\right)}. \end{aligned}$$

Now we use small angle approximation $\sin(x) = x$. Therefore,

$$\lim_{n\to\infty}\frac{1}{2^n\cdot\frac{\pi}{2^{n+1}}} = \frac{2}{\pi}.$$

Thus, the desired answer $\lfloor 1000\cdot\frac{2}{\pi}\rfloor = \boxed{636}$.

26. Jacob rolls 3^{2026} standard six-sided dice. For each integer r, let $p(r)$ be the probability that the sum of the values rolled is equivalent to r modulo 2026. From the set $\{0, 2, 4, \ldots, 2024\}$, Jacob chooses a and b for which $p(a)$ is minimal and $p(b)$ is maximal. Compute $a + b$.

Proposed by Darsh Patel

Answer: 1076

Let $\omega = \text{cis}\left(\frac{2\pi}{2026}\right)$. The generating function for the number of ways Jacob can roll a given sum is $(x + x^2 + x^3 + x^4 + x^5 + x^6)^{3^{2026}}$. Applying Roots of Unity Filter to $x^{-r}(x + x^2 + x^3 + x^4 + x^5 + x^6)^{3^{2026}}$, we find

$$\begin{aligned} 2026\cdot 6^{3^{2026}}\cdot p(r) &= \sum_{k=0}^{2025}\omega^{-kr}(\omega^{-k} + \omega^{-2k} + \omega^{-3k} + \omega^{-4k} + \omega^{-5k} + \omega^{-6k})^{3^{2026}} \\ &= 6^{3^{2026}} + \left(\sum_{k=1}^{1012}\omega^{-kr}(\omega^{-k} + \cdots + \omega^{-6k})^{3^{2026}} + \omega^{kr}(\omega^{k} + \cdots + \omega^{6k})^{3^{2026}}\right) \\ &= 6^{3^{2026}} + 2\left(\sum_{k=1}^{1012}\Re\left(\omega^{kr}(\omega^{k} + \cdots + \omega^{6k})^{3^{2026}}\right)\right) \end{aligned}$$

We will find the biggest terms in this sum. First let $\alpha = \text{cis}\left(\frac{\pi}{2026}\right)$, so that $\omega = \alpha^2$. Now

$$\begin{aligned} &|\omega^k + \omega^{2k} + \omega^{3k} + \omega^{4k} + \omega^{5k} + \omega^{6k}| \\ =&|\alpha^{2k} + \alpha^{4k} + \alpha^{6k} + \alpha^{8k} + \alpha^{10k} + \alpha^{12k}| \\ =&|\alpha^{-5k} + \alpha^{-3k} + \alpha^{-k} + \alpha^{k} + \alpha^{3k} + \alpha^{5k}| \\ =&2|\Re(\alpha^k) + \Re(\alpha^{3k}) + \Re(\alpha^{5k})| \\ =&2\left|\cos\left(\frac{k\pi}{2026}\right) + \cos\left(\frac{3k\pi}{2026}\right) + \cos\left(\frac{5k\pi}{2026}\right)\right| \end{aligned}$$

It is not hard to check that for $k = 1, \ldots, 1012$, the largest possible value of this expression occurs at $k = 1$, and the second largest at $k = 2$. We also have that

$$\begin{aligned}
&\left(\frac{\cos\left(\frac{2\pi}{2026}\right)+\cos\left(\frac{6\pi}{2026}\right)+\cos\left(\frac{10\pi}{2026}\right)}{\cos\left(\frac{\pi}{2026}\right)+\cos\left(\frac{3\pi}{2026}\right)+\cos\left(\frac{5\pi}{2026}\right)}\right)^{3^{2026}} \\
=&\left(1-\frac{\cos\left(\frac{\pi}{2026}\right)-\cos\left(\frac{2\pi}{2026}\right)+\cos\left(\frac{3\pi}{2026}\right)-\cos\left(\frac{6\pi}{2026}\right)+\cos\left(\frac{5\pi}{2026}\right)-\cos\left(\frac{10\pi}{2026}\right)}{\cos\left(\frac{\pi}{2026}\right)+\cos\left(\frac{3\pi}{2026}\right)+\cos\left(\frac{5\pi}{2026}\right)}\right)^{3^{2026}} \\
<&\left(1-\frac{\cos\left(\frac{5\pi}{2026}\right)-\cos\left(\frac{10\pi}{2026}\right)}{3}\right)^{3^{2026}} \\
<&(1-10^{-5})^{3^{2026}} \\
<&e^{-10^{-5}\cdot 3^{2026}} \\
<&e^{-3^{2014}}
\end{aligned}$$

This bound is strong enough to imply that when comparing values of $p(r)$, it suffices to consider first only the contribution from the $\Re\left(\omega^{kr}(\omega^k+\cdots+\omega^{6k})^{3^{2026}}\right)$ term at $k = 1$. This complex number has constant magnitude across all choices of r, and seeing as

$$\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = \alpha^7(\alpha^{-5}+\alpha^{-3}+\alpha^{-1}+\alpha+\alpha^3+\alpha^5)$$

has argument $\frac{7\pi}{2026}$, its argument is $\frac{(2r-7\cdot 3^{2026})\pi}{2026}$. Since $3^{2026} \equiv 9 \pmod{4052}$, this is just $\frac{(2r-63)\pi}{2026}$. The maximum value of $p(r)$ occurs when this is closest to 0 modulo 2π, which occurs uniquely at $r = 32$, and the minimum value occurs when this is closest to π modulo 2π, which occurs uniquely at $r = 1044$. The desired value is then $32 + 1044 = \boxed{1076}$.

27. A country contains the 5 cities A, B, C, D, and E. Between every pair of cities, a one-way road is built, whose direction is chosen at random. Let p be the probability that one can use the roads to travel from A to B, possibly traveling through intermediate cities. If p can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Benjamin Jiang

Answer: 44212

We will utilize complementary counting. Let the other 3 vertices be C, D, and E. Notice that there must be a directed edge from B to A, which happens with a probability of $\frac{1}{2}$. We will now utilize casework.

- Suppose there are 0 directed edges leading into B. This happens with a probability of $\frac{1}{2}\cdot\frac{1}{8} = \frac{1}{16}$. There obviously exists no path from A to B here.
- Suppose there is 1 directed edge leading into B. There is a $\frac{3}{16}$ chance of this happening. Without the loss of generality, suppose this directed edge come from C. There is a $\frac{1}{2}$ chance that the directed edge A to C does not exist.

  Without the loss of generality, suppose there be a directed edge from D to E. If the directed edge from A to D exists, then there must exist directed edges from C to D and from C to E. Else, there must exist a directed edge from C to E or from E to A (not necessarily both).

  This gives us a total probability of

$$\frac{3}{16}\cdot\frac{1}{2}\cdot\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\right) = \frac{3}{64}.$$

- Suppose there are 2 directed edges leading into B. There is a $\frac{3}{16}$ chance of this happening. Without the loss of generality, let those edges come from C and D.

  There must be edges leading from C and D into A, for a probability of $\frac{1}{4}$. From there, if the directed edge from A to E exists, there must be edges leading from C and D into E. If it does not exist, there does not exist a path from A to B. The probability associated with this is $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$.

  The probability for this case is thus

$$\frac{3}{16} \cdot \frac{1}{4} \cdot \frac{5}{8} = \frac{15}{512}.$$

- Suppose there are 3 directed edges leading into B. The probability of this case is $\frac{1}{16}$. There must be edges into A from C, D, E, for a probability of $\frac{1}{8}$. This means that the total probability for this case is $\frac{1}{128}$

Total probability of no path existing is therefore

$$\frac{1}{16} + \frac{3}{64} + \frac{15}{512} + \frac{1}{128} = \frac{75}{512}.$$

Hence, the probability of a path existing is $1 - \frac{75}{512} = \frac{437}{512}$. The answer is therefore, $\boxed{44212}$.

28. A nonzero polynomial $P(x)$ with integer coefficients with no integer roots is such that $P(n+5) - P(n)$ is not divisible by 25 for any integer n. For all positive integers n, let s_n be the largest nonnegative integer k such that 5^k divides $P(1)P(2)\dots P(n)$. Given that $s_{15625} = 7813$, compute the sum of all distinct possible values of s_{1250}.

Proposed by Darsh Patel

Answer: 8605

The first step is to use Hensel Lifting Lemma to lift the roots of $P(x)$ modulo powers of 5. Hensel Lemma states that if r is a root of $P(x)$ modulo 5^k such that $P'(r) \not\equiv 0 \pmod 5$, then $P(x)$ has a unique root modulo 5^{k+1} that is equivalent to r modulo 5^k. This is a result of the congruence

$$P(x + 5^k t) \equiv P(x) + 5^k t P'(x) \pmod{5^{k+1}}$$

which can be verified by expansion. The same calculation gives that $\frac{P(x+5)-P(x)}{5} \equiv P'(x) \pmod 5$; thus, we are given that $P'(x)$ has no roots modulo 5, and Hensel Lemma can always be applied to lift roots. While we can handle the behavior of $\nu_5(P(x))$ by lifting roots to modulo a large power of 5, the most convenient approach is to work directly in $\mathbb{Z}_5$, the ring of 5-adic integers.

Since $P'(x)$ has no roots modulo 5, the roots of $P(x)$ in $\mathbb{Z}/5\mathbb{Z}$ are distinct. As we noted before, by Hensel Lemma, each of the roots of $P(x)$ in $\mathbb{Z}/5\mathbb{Z}$ uniquely correspond to a root of $P(x)$ in $\mathbb{Z}_5$. Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be the roots of $P(x)$ in $\mathbb{Z}_5$. Write $P(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_k)Q(x)$ for some 5-adic integer polynomial $Q(x)$ having no roots modulo 5. Now for any n, we have

$$\begin{aligned}
s_n &= \nu_5(P(1)P(2)\dots P(n)) \\
&= \nu_5\left(\prod_{k=1}^{n}(k-\alpha_1)(k-\alpha_2)\dots(k-\alpha_s)\right) \\
&= \sum_{i=1}^{s}\nu_5\left(\prod_{k=1}^{n}(k-\alpha_i)\right) \\
&= \sum_{i=1}^{s}\nu_5\left(n! \cdot \binom{n-\alpha_i}{n}\right) \\
&= s\nu_5(n!) + \left(\sum_{i=1}^{s}\nu_5\left(\binom{n-\alpha_i}{n}\right)\right)
\end{aligned}$$

We can compute $\nu_5(n!)$ by Legendre Formula as $\nu_5(n!) = \frac{n-s_5(n)}{4}$. Also, we can express $\nu_5\left(\binom{n-\alpha_i}{n}\right)$ by Kummer Theorem as the number of carries when n is added to $-\alpha_i$. For $n = 15625 = 5^6$, we have $\nu_5(5^6!) = \frac{5^6-1}{4}$. Since $7813 = \frac{5^6+1}{2}$ lies strictly between 0 and 3 $\left(\frac{5^6-1}{4}\right)$, we must have $s = 1, 2$. In finding s_{1250}, we note $1250 = 2 \cdot 5^4$.

Case 1: $s = 1$
Then $\nu_5\left(\binom{5^6-\alpha_1}{5^6}\right)$, the number of carries when 5^6 is added to $-\alpha_1$, is $\frac{5^6+1}{2} - \frac{5^6-1}{4} = \frac{5^6+3}{4}$. Now when $2 \cdot 5^4$ is added to $-\alpha_1$, there could be

- 0 carries
- 1 carry to the 5^5 digit
- 2 carries to the 5^6 digit, which now forces an additional $\frac{5^6+3}{4}$ carries based on our analysis of $5^6 + (-\alpha_1)$

Since $\nu_5((2 \cdot 5^4)!) = \frac{5^4-1}{2}$, we have

$$s_{1250} \in \left\{0, 1, 2 + \frac{5^6+3}{4}\right\} + \frac{5^4-1}{2}$$

Case 2: $s = 2$
Then $\nu_5\left(\binom{5^6-\alpha_1}{5^6}\right) + \nu_5\left(\binom{5^6-\alpha_2}{5^6}\right)$ is $\frac{5^6+1}{2} - 2\left(\frac{5^6-1}{4}\right) = 1$. WLOG assume $5^6 + (-\alpha_1)$ has 1 carry and $5^6 + (-\alpha_2)$ has 0. Now when $2 \cdot 5^4$ is added to $-\alpha_1$ and $-\alpha_2$, there are at most $2 + 1 = 3$ carries in $-\alpha_1$ and 2 carries in $-\alpha_2$. Each of the possibilities from 0 total carries to $3 + 2 = 5$ are achieved by

- 0 carries in $-\alpha_1$ and 0 carries in $-\alpha_2$
- 0 carries $-\alpha_1$ and 1 carry in $-\alpha_2$
- 0 carries $-\alpha_1$ and 2 carries in $-\alpha_2$
- 2 carries in $-\alpha_1$, which forces an additional 1 carry, and 0 carries in $-\alpha_2$
- 2 carries in $-\alpha_1$, which forces an additional 1 carry, and 1 carry in $-\alpha_2$
- 2 carries in $-\alpha_1$, which forces an additional 1 carry, and 2 carries in $-\alpha_2$

Then we have

$$s_{1250} \in \{0, 1, 2, 3, 4, 5\} + 2\left(\frac{5^4-1}{2}\right)$$

Now the possibilities for s_{1250} are

$$\left\{\frac{5^4-1}{2}, \frac{5^4+1}{2}, 5^4-1, 5^4, 5^4+1, 5^4+2, 5^4+3, 5^4+4, \frac{5^4+3}{2} + \frac{5^6+3}{4}\right\}$$

The requested sum is $\boxed{8605}$.

It remains to show that the necessary $P(x)$ always exists. We first show that such $P(x)$ exists in $\mathbb{Z}_5$. It is easy to see that the necessary $\{-\alpha_1\}$ or $\{-\alpha_1, -\alpha_2\}$ with desired carries always exist, so we need to show that we can always force $P'(x)$ to have no roots modulo 5 by choosing $Q(x)$. Take the $-\alpha_i$ to be any distinct values modulo 5, which is possible since we never need their 5^0 digits and since $s \leq 2 \leq 5$. Let $R(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_s)$. By Chain Rule, we have $P'(x) = R'(x)Q(x) + R(x)Q'(x)$. Since the roots of $R(x)$ are distinct modulo 5, we never have that $R(x)$ and $R'(x)$ are simultaneously 0. We can then choose values of $Q(x)$ and $Q'(x)$ with $Q(x)$ nonzero for each x modulo 5 so that $P'(x)$ is always nonzero, and then find some $Q(x)$ obtaining those values.

We now need to show that $P(x)$ exists as an integer polynomial with no integer roots. Since we only ever need finitely many digits of the $-\alpha_i$, it suffices to find a sufficiently strong 5-adic approximation of $P(x)$ in

integers. We first truncate the 5-adic expansions of the coefficients of $P(x)$ so they retain sufficiently many digits. This yields an integer polynomial, which we now need to force to have no integer roots. This can be fixed by adding $5^a(x^{2d}+b)$ to $P(x)$ for some $d > \frac{\deg(P)}{2}$, a sufficiently large so that the new polynomial is a sufficiently strong approximation of the original in 5-adic, and b sufficiently large so that the new polynomial is always positive.

29. Let $N = 10^{10}$. Integers $a_0, a_1, \ldots, a_N$ are such that

$$a_0 n^{2025} + a_1 m(n+1)^{2025} + a_2 m^2 (n+2)^{2025} + \cdots + a_N m^N (n+N)^{2025} = 0$$

for all positive integers m and n with $m \leq 1000$. Let s be the smallest nonzero possible value of $|a_0 - a_1 + a_2 + \cdots + (-1)^N a_N|$. Compute the largest nonnegative integer k such that 2^k divides s.

Proposed by Darsh Patel

Answer: 2013844

Fixing m, we have

$$a_0 x^{2025} + a_1 m(x+1)^{2025} + \cdots + a_N m^N (x+N)^{2025} = 0$$

for infinitely many x, which gives the above equation as a polynomial identity.

The key claim is that for any sequence $(a_0, a_1, \ldots)$ which is eventually always zero, we have $a_0 x^{2025} + a_1 m(x+1)^{2025} + a_2 m^2 (x+2)^{2025} + \cdots = 0$ if and only if $(m-x)^{2026}$ divides the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots$. This follows from the following:

Claim: For any polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and polynomial $P(x)$, the degree of $a_0 P(x) + a_1 P(x+1) + a_2 P(x+2) + \cdots$ is $\deg(P) - t$, where t is the multiplicity of $1-x$ as a factor of $A(x)$.
Here, it is understood that a polynomial of negative degree is the zero polynomial. We prove this statement by induction on t. In the base case of $t = 0$, we see that from $P(x)$ to $a_0 P(x) + a_1 P(x+1) + a_2 P(x+2) + \cdots$, the coefficient of $x^{\deg(P)}$ gets multiplied by $A(1) \neq 0$ and thus remains nonzero, and so the degree does not change. Now assume $t > 0$, and that we have already proved this statement for $t-1$. Write $A(x) = (1-x)B(x)$ and $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$. Then

$$\begin{aligned} a_0 P(x) + a_1 P(x+1) + a_2 P(x+2) + \cdots &= b_0 P(x) + (b_1 - b_0) P(x+1) + (b_2 - b_1) P(x+2) + \cdots \\ &= b_0 \Delta P(x) + b_1 \Delta P(x+1) + b_2 \Delta P(x+2) + \ldots \end{aligned}$$

Now applying the induction hypothesis, since $1-x$ divides $B(x)$ with multiplicity $t-1$, this polynomial has degree $\deg(\Delta P) - (t-1) = (\deg(P) - 1) - (t-1) = \deg(P) - t$. This completes the induction.

Let $A(x) = a_0 + a_1 x + \cdots + a_N x^N$. Applying our claim, the given condition is equivalent to $(1-x)^{2026}$ dividing $A(mx)$ for $m = 1, 2, \ldots, 1000$. Equivalently, we need that $(m-x)^{2026} = m^{2026}\left(1 - \frac{x}{m}\right)^{2026}$ divides $A(x)$. Thus, the condition becomes that $A(x)$ is divisible by

$$T(x) = (1-x)^{2026}(2-x)^{2026} \ldots (1000-x)^{2026}$$

Then $|a_0 - a_1 + a_2 - \cdots + (-1)^N a_N| = |A(-1)|$ is divisible by $T(-1) = (1001!)^{2026}$, and thus at least this value when nonzero. This is also achievable by taking $A(x) = T(x)$, since $N \geq \deg(T)$. The desired value is then $\nu_2((1001!)^{2026}) = \boxed{2013844}$.

30. Let $I = \displaystyle\int_1^{\sqrt{2}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\, dx$. Compute $\lfloor I^{-2} \rfloor$.

Proposed by Yusuf Sheikh

Answer: 94

We can note that, by IBP with $u = \arctan(\sqrt{2-x^2})$ and $dv = \frac{1}{1+x^2}dx$, we have

$$I = \int_1^{\sqrt{2}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\,\mathrm{d}x = -\frac{\pi^2}{16} + \int_1^{\sqrt{2}} \frac{x\arctan(x)}{(3-x^2)\sqrt{2-x^2}}\,\mathrm{d}x.$$

But then substituting $u = \sqrt{2-x^2}$ (with $du = \frac{x}{\sqrt{2-x^2}}\,dx$), we obtain

$$\int_1^{\sqrt{2}} \frac{x\arctan(x)}{(3-x^2)\sqrt{2-x^2}}\,\mathrm{d}x = \int_0^1 \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\,\mathrm{d}x.$$

Thus, we have

$$2I = -\frac{\pi^2}{16} + \int_0^{\sqrt{2}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\,\mathrm{d}x.$$

We now evaluate $\int_0^{\sqrt{2}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\,\mathrm{d}x$. The calculation is straightforward:

$$\begin{aligned}
\int_0^{\sqrt{2}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}\,\mathrm{d}x &\overset{x=\sqrt{2}\sin\theta}{=} \int_0^{\frac{\pi}{2}} \frac{\arctan(\sqrt{2}\cos\theta)\sqrt{2}\cos\theta}{1+2\sin^2\theta}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{2\cos^2\theta}{(1+2\sin^2\theta)(1+2t^2\cos^2\theta)}\,\mathrm{d}t\,\mathrm{d}\theta \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{2\cos^2\theta}{(1+2\sin^2\theta)(1+2t^2\cos^2\theta)}\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{2\sec^2\theta}{(\sec^2\theta+2\tan^2\theta)(\sec^2\theta+2t^2)}\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \int_0^1 \int_0^{\infty} \frac{2}{(3x^2+1)(x^2+2t^2+1)}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^1 \frac{1}{3t^2+1}\left(\int_0^{\infty} \frac{3}{3x^2+1} - \frac{1}{x^2+2t^2+1}\,\mathrm{d}x\right)\mathrm{d}t \\
&= \int_0^1 \frac{1}{3t^2+1}\left[\frac{\pi\sqrt{3}}{2} - \frac{\pi}{2\sqrt{2t^2+1}}\right]\mathrm{d}t \\
&= \frac{\pi^2}{6} - \frac{\pi}{2}\int_0^1 \frac{1}{(3t^2+1)\sqrt{2t^2+1}}\,\mathrm{d}t \\
&= \frac{\pi^2}{6} - \frac{\pi}{2}\left[\arctan\left(\frac{t}{\sqrt{2t^2+1}}\right)\right]_0^1 \\
&= \frac{\pi^2}{6} - \frac{\pi^2}{12} \\
&= \frac{\pi^2}{12}.
\end{aligned}$$

(For those not comfortable with double integrals, one can consider $F(t) = \int_0^{\frac{\pi}{2}} \frac{\arctan(t\sqrt{2}\cos\theta)\sqrt{2}\cos\theta}{1+2\sin^2\theta}\,\mathrm{d}\theta$ and then perform Feynman's trick by computing $F'(t)$ then $\int_0^1 F'(t)$ in the usual way; this will be done in the exact manner as the above calculation). So, we conclude

$$2I = -\frac{\pi^2}{16} + \frac{\pi^2}{12} = \frac{\pi^2}{48} \iff I = \frac{\pi^2}{96}.$$

The desired value is $\left\lfloor \left(\frac{\pi^2}{96}\right)^{-2} \right\rfloor = \boxed{94}$.