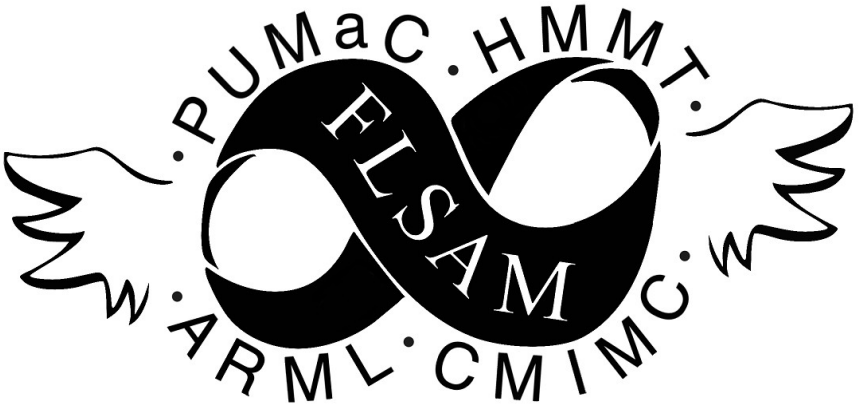
Florida Student Association of Mathematics



2026 HMMT Tryout

January 2026

Solutions

Algebra and Number Theory Answers

1. 45
2. $\frac{55}{2}$
3. $\frac{135}{4}$
4. 4851
5. $\frac{929}{100}$
6. 563

Geometry Answers

1. 69
2. $94\sqrt{3}$
3. $72\sqrt{3}$
4. $20\sqrt{5}$
5. 15
6. $405\sqrt{7}$

Combinatorics Answers

1. $\frac{576}{1925}$
2. $\frac{35}{72}$
3. 4094
4. 596
5. $\frac{1861}{512}$
6. $10 - \pi^2$

Algebra and Number Theory

1. There exists a unique positive integer n whose positive divisors have arithmetic mean 13. Find n.

Proposed by Aaryan Vaishya & Darsh Patel

Answer: 45

Let $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$. Then the average divisor of n is

$$\frac{\sigma(n)}{\tau(n)} = \left(\frac{1 + p_1 + p_1^2 + \cdots + p_1^{e_1}}{e_1 + 1}\right)\left(\frac{1 + p_2 + p_2^2 + \cdots + p_2^{e_2}}{e_2 + 1}\right) \cdots \left(\frac{1 + p_n + p_n^2 + \cdots + p_n^{e_n}}{e_n + 1}\right)$$

Among small prime powers, the clearest candidate for producing a factor of 13 in the numerator of this expression is 3^2, which yields a term of $\frac{13}{3}$. We now note that 5^1 produces $\frac{6}{2}$, and so $3^2 \cdot 5^1 = \boxed{45}$ works.

A full proof that this is the only solution can be carried out by lower bounding $\frac{\sigma(n)}{\tau(n)}$ and performing a finite case check.

2. Quadratic polynomials $P(x)$ and $Q(x)$ with real coefficients are such that the polynomials $P(x)$, $Q(x)$, and $P(x) + Q(x)$ achieve unique minimum values when x is 20, 26, and 25 respectively. Find the x for which $P(x) - Q(x)$ is maximized.

Proposed by Jacob Khohayting

Answer: $\frac{55}{2}$

We ignore constant terms. Define $P(x) = Ax^2 - 40Ax$ and $Q(x) = Dx^2 - 52Dx$. Then,

$$-\frac{1}{2} \cdot \frac{-40A - 52D}{A + D} = 25 \implies 5A = D.$$

Thus,

$$x = -\frac{1}{2} \cdot \frac{-40A + 52D}{A - D} = -\frac{1}{2} \cdot \frac{-40A + 260A}{A - 5A} = -\frac{1}{2} \cdot \frac{220A}{-4A} = \boxed{\frac{55}{2}}.$$

The initial problem which asked for the x where $P(x) - Q(x)$ is minimized is flawed. This is because $P(x) - Q(x)$ achieves a maximum value instead of a minimum value since it has negative leading coefficient. This problem has been edited for practice purposes.

3. Distinct reals a, b, and c satisfy

$$\begin{aligned} a + b &= \sqrt[3]{3a + 3b - 20} + 7 \\ b + c &= \sqrt[3]{3b + 3c - 20} + 7 \\ c + a &= \sqrt[3]{3c + 3a - 20} + 7 \end{aligned}$$

Compute $ab + bc + ca$.

Proposed by Jacob Khohayting

Answer: $\frac{135}{4}$

Notice, $a+b$, $b+c$, and $c+a$ are roots of the polynomial:

$$\begin{aligned} x = \sqrt[3]{3x-20}+7 &\iff (x-7)^3 = 3x-20 \\ &\iff x^3-21x^2+144x-323=0. \end{aligned}$$

By Vieta's, $2a+2b+2c=21$, so $a+b+c=\frac{21}{2}$. Thus, we may compute

$$\begin{aligned} ab+bc+ca &= (a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)-(a+b+c)^2 \\ &= 144-\left(\frac{21}{2}\right)^2 \\ &= \boxed{\frac{135}{4}}. \end{aligned}$$

The initial problem involving substitutions with the equation $x = \sqrt[3]{2x-11}+4$ is flawed because the cubic $x^3-12x^2+46x-53$ only has one real root. In order to make the problem use-able for future use, the problem has been adjusted for practice purposes to involve the function $x=\sqrt[3]{3x-20}+7$, or upon expansion $f(x)=x^3-21x^2+144x-323$, which can be verified to have three distinct real roots.

4. Let S be the set of ordered quadruples (a,b,c,p) of positive integers for which there exists a nondegenerate triangle with side lengths a, b, and c and with perimeter p such that $p<200$. Compute $\left\lfloor \sum_{(a,b,c,p)\in S} 2^{p-200} \right\rfloor$.

Proposed by Darsh Patel

Answer: 4851

We use Ravi substitution to express the condition that a, b, and c are the sides of a triangle by

$$\begin{aligned} x &= -a+b+c \\ y &= a-b+c \\ z &= a+b-c \end{aligned}$$

all being positive. We can then work with (x,y,z) instead of (a,b,c), where we need the condition $x\equiv y\equiv z \pmod 2$. Then $p=x+y+z$. Now for a fixed p, we count the number of (x,y,z) with $x+y+z=p$. Note that we have $x,y,z\equiv p \pmod 2$. If $p\equiv 0 \pmod 2$, then the number of solutions to $\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=\frac{p}{2}$ is $\binom{\frac{p}{2}-1}{2}$. If $p\equiv 1 \pmod 2$, then the number of solutions to $\frac{x+1}{2}+\frac{y+1}{2}+\frac{z+1}{2}=\frac{p+3}{2}$ is $\binom{\frac{p+1}{2}}{2}$. The contribution from $p=2k, 2k+1$ to the sum $\sum_{(a,b,c,p)\in S} 2^p$ is then

$$\binom{k-1}{2}2^{2k}+\binom{k+1}{2}2^{2k+1}=\left(\frac{3k^2-k+2}{2}\right)4^k$$

To sum this from $k=1$ to 99, we look for a telescope. We have the finite differences

$$\begin{aligned} (k+1)^2 4^{k+1}-k^2 4^k &= (3k^2+8k+4)4^k \\ (k+1)4^{k+1}-k4^k &= (3k+4)4^k \\ 4^{k+1}-4^k &= (3)4^k \end{aligned}$$

Then we can find

$$\left(\frac{3k^2-k+2}{2}\right)4^k=\left(\frac{(k+1)^2-3(k+1)+\frac{10}{3}}{2}\right)4^{k+1}-\left(\frac{k^2-3k+\frac{10}{3}}{2}\right)4^k$$

Telescoping yields

$$\sum_{(a,b,c,p)\in S} 2^p = \left(\frac{100^2 - 3(100) + \frac{10}{3}}{2}\right) 4^{100} - \frac{8}{3} = \left(4851 + \frac{2}{3}\right) 2^{200} - \frac{8}{3}$$

Then $\left\lfloor \sum_{(a,b,c,p)\in S} 2^{p-200} \right\rfloor = 4851.$

5. Let P be the product of all primes less than 20. A divisor d of P^{99} is randomly chosen. Compute the expected value of $\nu_2(\sigma(d))$.

[For a positive integer n, $\sigma(n)$ denotes the sum of its divisors and $\nu_2(n)$ denotes the largest k for which 2^k divides n.]

Proposed by Darsh Patel

Answer: $\frac{929}{100}$

Let $p_1 < p_2 < \cdots$ be the primes, so that $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (2, 3, 5, 7, 11, 13, 17, 19)$. Write $d = p_1^{e_1} p_2^{e_2} \ldots p_8^{e_8}$, where $e_1, e_2, \ldots, e_8 \leq 99$. We have

$$\sigma(d) = \left(\frac{p_1^{e_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{e_2+1} - 1}{p_2 - 1}\right) \cdots \left(\frac{p_8^{e_8+1} - 1}{p_8 - 1}\right)$$

Clearly $\nu_2\left(\frac{2^{e_1+1}-1}{2-1}\right) = 0$. We will compute the 2-adic valuation of the rest of the terms using Lifting the Exponent Lemma. If $p \equiv 1 \pmod 4$, then

$$\nu_2\left(\frac{p^{e+1} - 1}{p - 1}\right) = (\nu_2(p-1) + \nu_2(e+1)) - \nu_2(p-1) = \nu_2(e+1)$$

The expected value of $\nu_2\left(\frac{p^{e+1}-1}{p-1}\right)$ is then $\frac{\nu_2(100!)}{100}$. If $p \equiv 3 \pmod 4$, we have that $\frac{p^{e+1}-1}{p-1} \equiv 1 \pmod 2$ when $e+1$ is odd and

$$\nu_2\left(\frac{p^{e+1} - 1}{p - 1}\right) = \nu_2((p^2)^{\frac{e+1}{2}} - 1) - \nu_2(p-1) = \left(\nu_2(p^2-1) + \nu_2\left(\frac{e+1}{2}\right)\right) - \nu_2(p-1) = \nu_2(e+1) + \nu_2\left(\frac{p+1}{2}\right)$$

when $e+1$ is even. The expected value of $\nu_2\left(\frac{p^{e+1}-1}{p-1}\right)$ is then $\frac{\nu_2(100!)}{100} + \frac{1}{2}\nu_2\left(\frac{p+1}{2}\right)$. The total expected value is then

$$7\left(\frac{\nu_2(100!)}{100}\right) + \frac{1}{2}\left(\nu_2\left(\frac{3+1}{2}\right) + \nu_2\left(\frac{7+1}{2}\right) + \nu_2\left(\frac{11+1}{2}\right) + \nu_2\left(\frac{19+1}{2}\right)\right) = 7\left(\frac{97}{100}\right) + \frac{5}{2} = \boxed{\frac{929}{100}}$$

6. There exists a unique prime p such that $n^4 + n^3 + n^2 + n - 1$ is divisible by p for exactly 3 integer values of n between 1 and p inclusive. Find p.

Proposed by Darsh Patel

Answer: 563

Factoring out the 3 distinct roots from our quartic, we will be left with a monic linear polynomial, which clearly has a root. Then this root must be the same as one we already factored out, and thus the quartic has

a double root. Let r be the double root of $f(x) = x^4 + x^3 + x^2 + x - 1$ modulo p. Since $(x-r)^2$ divides $f(x)$ modulo p, we also have $f'(r) \equiv 0$. Performing Euclidean Algorithm, we obtain

$$\begin{aligned} r^4 + r^3 + r^2 + r - 1 &\equiv 0 \\ 4r^3 + 3r^2 + 2r + 1 &\equiv 0 \\ r^3 + 2r^2 + 3r - 4 \equiv 4(r^4 + r^3 + r^2 + r - 1) - r(4r^3 + 3r^2 + 2r + 1) &\equiv 0 \\ 5r^2 + 10r - 17 \equiv 4(r^3 + 2r^2 + 3r - 4) - (4r^3 + 3r^2 + 2r + 1) &\equiv 0 \\ 32r - 20 \equiv 5(r^3 + 2r^2 + 3r - 4) - r(5r^2 + 10r - 17) &\equiv 0 \\ -\frac{563}{64} \equiv 5\left(\frac{20}{32}\right)^2 + 10\left(\frac{20}{32}\right) - 17 &\equiv 0 \end{aligned}$$

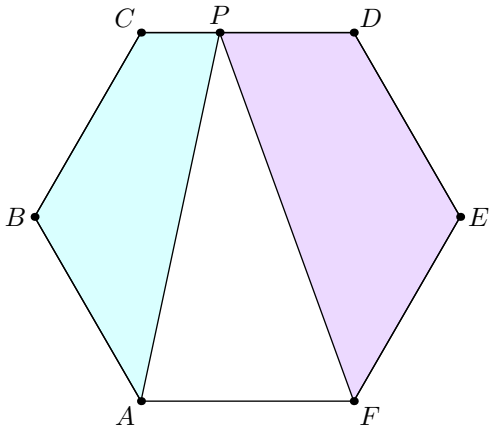
The last line is valid since we clearly have $p \neq 2$. Since 563 is prime, we must have $p = \boxed{563}$. We remark that one can factor $x^4 + x^2 + x^2 + x - 1 \equiv (x-71)^2(x-199)(x-221) \pmod{563}$.

Geometry

1. Let $ABCDEF$ be a regular hexagon. Suppose P lies on side CD such that $PABC$ and $PDEF$ have areas 20 and 26, respectively. Compute the area of $ABCDEF$.

Proposed by Jacob Khohayting

Answer: 69



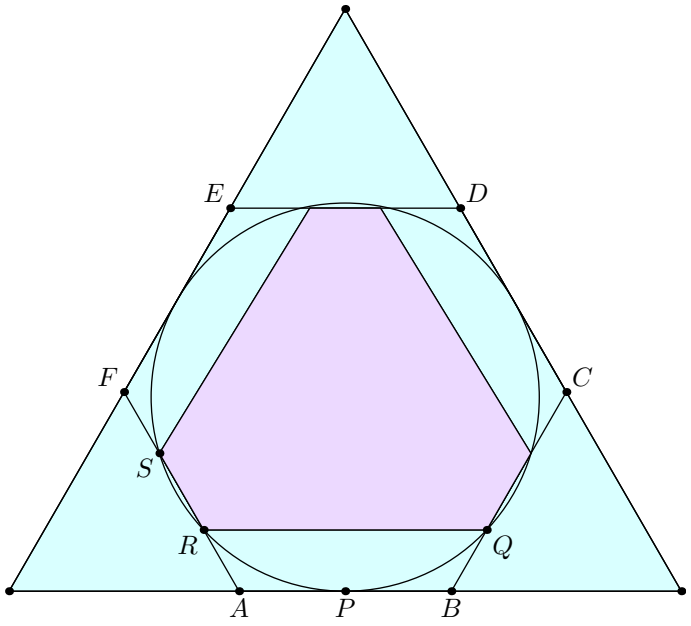
Let $[ABCDEF] = S$. Then, $[PAF] = \frac{S}{3}$. Thus,

$$S = [ABCDEF] = [PABC] + [PDEF] + [PAF] = 20 + 26 + \frac{S}{3} \iff S = \boxed{69}.$$

2. Let $ABCDEF$ be an equiangular hexagon with alternating sides of lengths 8 and 10. A circle, tangent to the sides of length 8, intersects the sides of length 10 at six distinct points. Compute the area of the polygon formed by the six intersection points.

Proposed by Jacob Khohayting

Answer: $94\sqrt{3}$



Let AF intersect the circle at R and S with $AR < AS$, and BC intersect the circle at Q such that $BQ < CQ$. The total area of $ABCDEF$ is

$$\frac{(10+8+10)^2\sqrt{3}}{4} - 3 \cdot \frac{10^2\sqrt{3}}{4} = \frac{484\sqrt{3}}{4} = 121\sqrt{3}.$$

Let $AB = CD = EF = 8$ and $BC = DE = FA = 10$. By power of point at A, then:

$$AR \cdot AS = AP^2 \implies AR \cdot (10 - AR) = 4^2 \implies AR = 2, 8.$$

Clearly, we require $AR = 2$. Then, $RQ = AR + AB = 10$. We will compute the area of trapezoid $RABQ$:

$$[RABQ] = \frac{AB + RQ}{2} \cdot d(AB, RQ) = \frac{8+10}{2} \cdot \sqrt{3} = 9\sqrt{3}.$$

Therefore, the desired area is $[ABCDEF] - 3 \cdot [RABQ] = 121\sqrt{3} - 3 \cdot 9\sqrt{3} = \boxed{94\sqrt{3}}$.

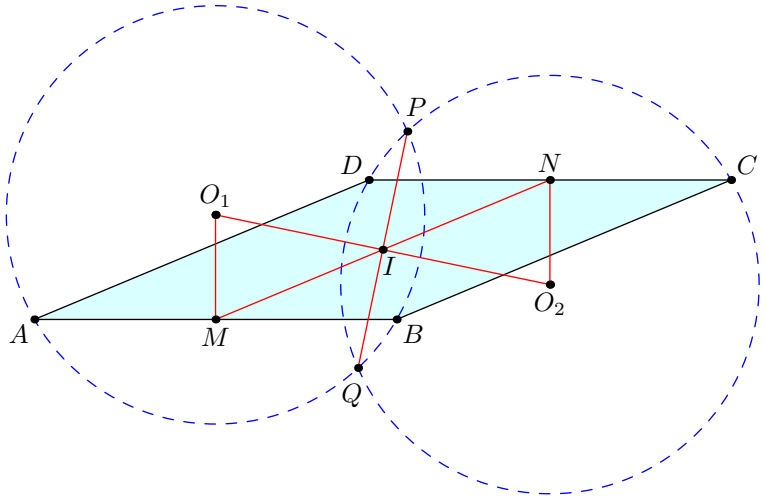
3. Let $ABCD$ be a rhombus of side length 18. Points P and Q lie outside $ABCD$ such that

$$\angle APB = \angle CQD = 60^\circ \quad \text{and} \quad \angle AQB = \angle CPD = 120^\circ.$$

Given that $PQ = 12$, compute the area of $ABCD$.

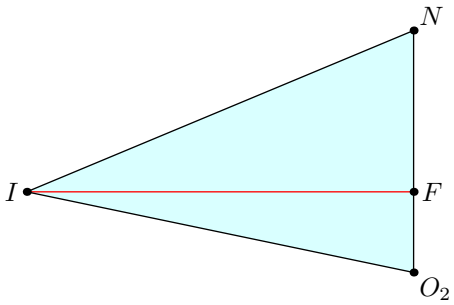
Proposed by Jacob Khohayting

Answer: $72\sqrt{3}$



Firstly, $MN = AD = 18$. Thus, $IN = 9$. As $ABCD$ has a side length of 18, then the radius of each circle is $\frac{18}{\sqrt{3}} = 6\sqrt{3}$. As $PQ = 12$, then $IP = 6$. So, $IO_2 = 6\sqrt{2}$. Lastly, $NO_2 = 3\sqrt{3}$.

Therefore, consider $\triangle NIO_2$ with $NI = 9$, $IO_2 = 6\sqrt{2}$, and $O_2N = 3\sqrt{3}$. Let F be the perpendicular from I to NO_2, as shown below.



We wish to compute NF to determine the height. By Pythagoras,

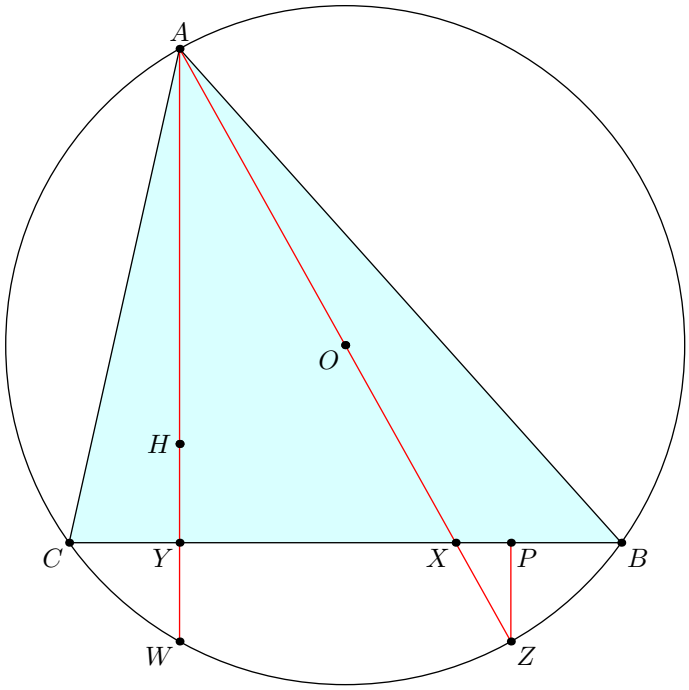
$$NI^2 - NF^2 = O_2I^2 - O_2F^2 \implies 9^2 - NF^2 = (6\sqrt{2})^2 - (3\sqrt{3} - NF)^2.$$

Solving, then $NF = 2\sqrt{3}$. Thus, the height of the rhombus is $4\sqrt{3}$. Hence, $[ABCD] = 18 \cdot 4\sqrt{3} = \boxed{72\sqrt{3}}$.

4. Let ABC be an acute triangle with orthocenter H and circumcenter O. Suppose lines AO and AH intersect segment BC at X and Y, respectively, such that $BX = 3$, $XY = 5$, and $YC = 2$. Compute the area of $\triangle ABC$.

Proposed by Jacob Khohayting

Answer: $20\sqrt{5}$



Let AH and AO intersect (ABC) at W and Z. Then, suppose P is a point on BC such that $ZP \perp BC$.

Note that Y and W are reflections of P and Z across the perpendicular bisector of BC. Therefore, $CY = PB$. Hence, $XP = 1$. But, $\triangle AYX \sim \triangle XPZ$. Therefore, if $YW = PZ = t$, then $AY = 5t$. So, by Power of Point with respect to Y, then:

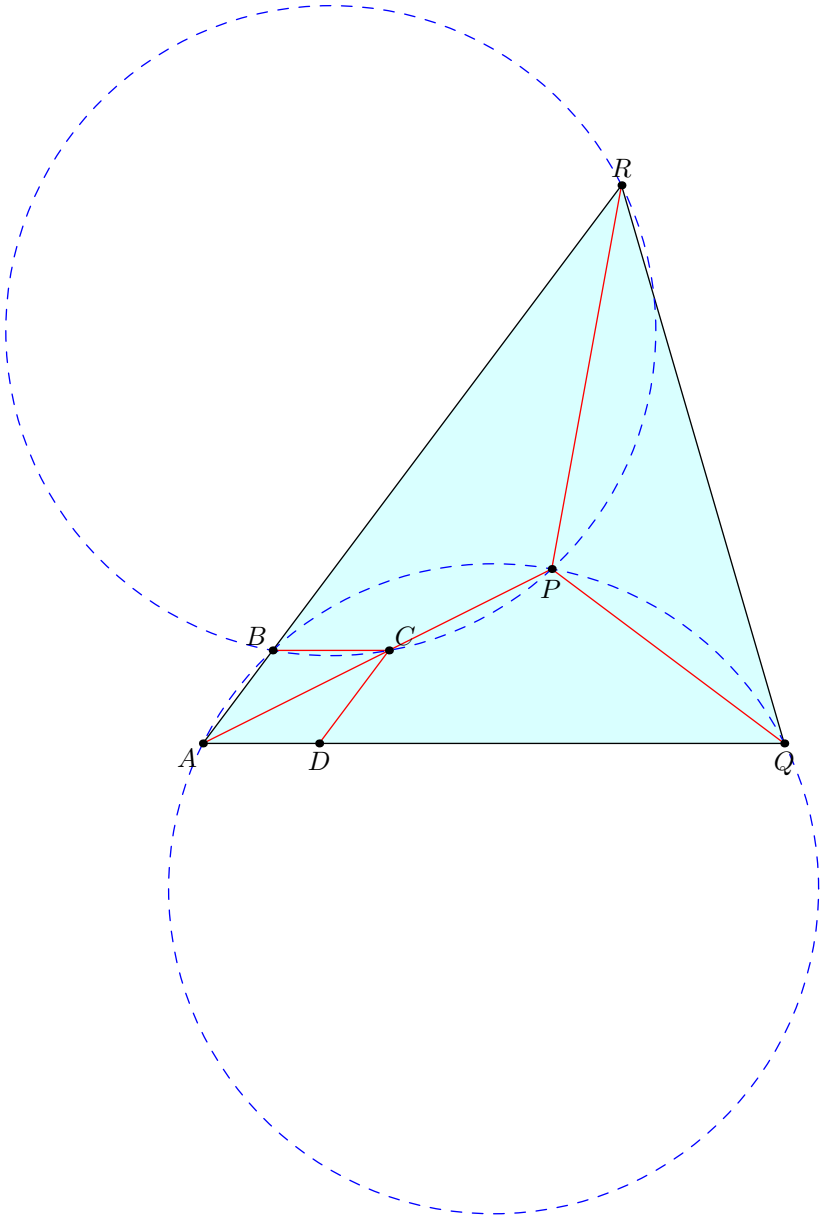
$$AY \cdot YW = CY \cdot YB \implies 5t^2 = 16 \implies t = \tfrac{4}{5}\sqrt{5}.$$

Thus, $AY = 5t = 4\sqrt{5}$. Therefore, $[ABC] = \frac{1}{2} \cdot AY \cdot BC = \frac{1}{2} \cdot 4\sqrt{5} \cdot 10 = \boxed{20\sqrt{5}}$.

5. In rhombus $ABCD$ with $\angle A < \angle B$, extend AC past C to P. The circumcircles of ABP and BCP meet lines AD and AB at Q and R, respectively. If $BR = 50$, $CP = 7\sqrt{5}$, and $DQ = 40$, compute the distance from P to RQ.

Proposed by Jacob Khohayting

Answer: 15



Note that AC bisects $\angle A$. Hence, $BP = PQ$. Then,

$$\angle BRP = 180 - \angle BCP = \angle BCA = \angle CAD = \angle PAQ.$$

By Extended Law of Sines,

$$2R_{(ABP)} = \frac{PQ}{\sin(\angle PAQ)} = \frac{PB}{\sin(\angle PRB)} = 2R_{(BCP)}.$$

Hence, $(BCP) \cong (ABP)$.

Now,

$$\angle PQA = 180 - \angle PBA = \angle PBR.$$

So, $RP = PA$. Thus, $\triangle RPB \cong \triangle APQ$. Hence, $RB = AQ$, so $AD = 10$.

By Power of Point at A, then

$$AB \cdot AR = AC \cdot AP \iff 10 \cdot 60 = AC \cdot \left(AC + 7\sqrt{5}\right) \iff AC = 8\sqrt{5}.$$

Then, $\cos(A) = \frac{3}{5}$. By Law of Cosines, then

$$RQ^2 = AR^2 + AQ^2 - 2 \cdot AR \cdot AQ \cdot \cos(\angle RAQ) = 60^2 + 50^2 - 2 \cdot 60 \cdot 50 \cdot \tfrac{3}{5} = 50^2.$$

Hence, $RQ = BR$. But, $BP = PQ$. So, RP bisects $\angle BRQ$. Thus, P is the incenter.

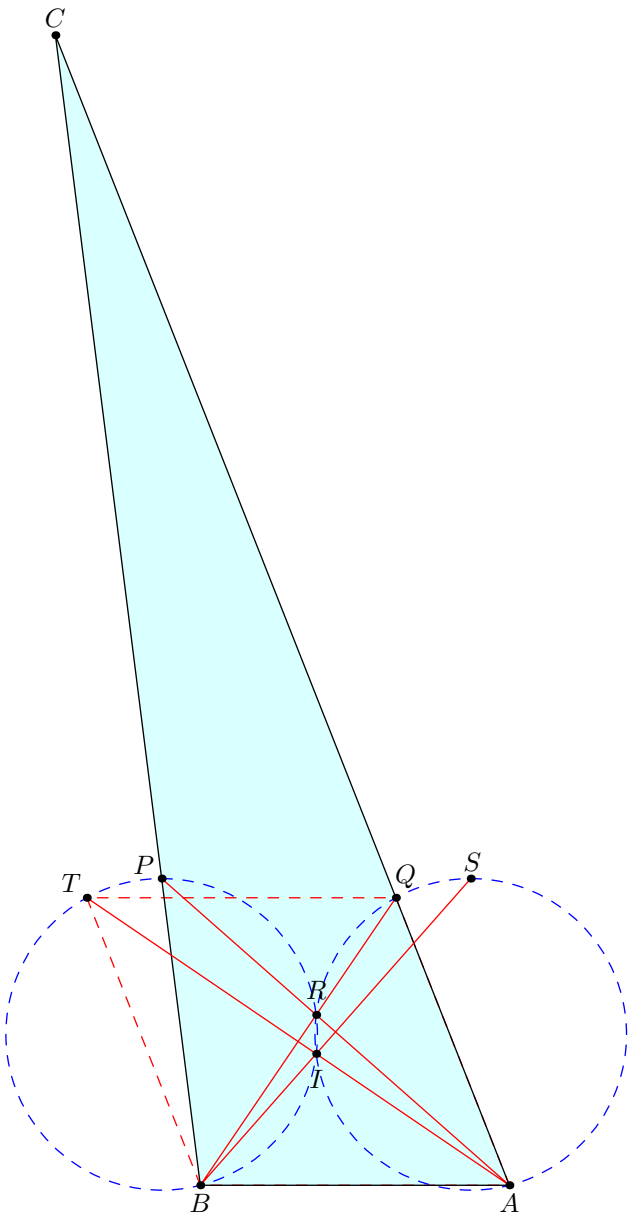
Now, $\triangle ARQ$ has side lengths 50, 50, and 60 yielding an area of 1200. So, we may compute

$$d(P, RQ) = \frac{2 \cdot [ARQ]}{AR + RQ + QA} = \frac{2 \cdot 1200}{50 + 50 + 60} = \boxed{15}.$$

6. Let ABC be a triangle with $AB = 24$. Points P and Q lie on segments CA and CB such that $PA = AB = BQ$. Suppose AQ and BP intersect at R. Points S and T lie on the circumcircles of ARP and BRQ such that $AS \parallel CB$ and $BT \parallel CA$. Given that $SR = 16$ and $RT = 20$, compute the area of ABC.

Proposed by Jacob Khohayting

Answer: $405\sqrt{7}$



Let I be the incenter of ABC. Then, I is the orthocenter of ABR.

Now,

$$\angle IRB = \angle IAB = \angle IPB.$$

So, I lies on (BRP).

Now,

$$\begin{aligned}
\angle TIB + \angle BIA &= (180^\circ - \angle ITB - \angle IBT) + \angle BIA \\
&= (180^\circ - \angle IRB - (\angle TBA - \angle IBA)) + \angle BIA \\
&= (180^\circ - \angle IRB - \angle TBA + \angle IBA) + \angle BIA \\
&= (180^\circ - \angle IRB - (180 - \angle CAB) + \angle IBA) + \angle BIA \\
&= (\angle CAB - \angle IRB + \angle IBA) + \angle BIA \\
&= (2\angle IAB - \angle IAB + \angle IBA) + \angle BIA \\
&= \angle IAB + \angle IBA + \angle BIA \\
&= 180^\circ.
\end{aligned}$$

Thus, T, I, and A are collinear.

Now,

$$\begin{aligned}
\angle PTB - \angle TPB &= \angle ARB - \angle TIB \\
&= (180^\circ - \angle AIB) - (180^\circ - \angle AIB) \\
&= 0^\circ.
\end{aligned}$$

Thus, PTB is isosceles.

So, as $TB = PB = BA = AQ$ and $TB \parallel QA$, then $TBQA$ is a rhombus. Note that R lies on diagonal BQ, then $AR = RT = 20$. Similarly, $BR = 16$.

Now, suppose that F is the foot of the perpendicular from I to AB. By Pythagorean Theorem, $FB = 9$, $FR = 5\sqrt{7}$, and $FA = 15$. So, due to the Orthocenter Reflection Lemma,

$$d(I, AB) = \frac{FB \cdot FA}{FR} = \frac{9 \cdot 15}{5\sqrt{7}} = \frac{27}{\sqrt{7}}.$$

Note that this implies $\frac{27}{\sqrt{7}}$ is the inradius. Let s denote the semi-perimeter of ABC. Let G be the foot of the perpendicular from I to BC. Suppose $CG = s$. The semiperimeter is $s + 24$. Then, we compute the area in two ways:

$$[ABC] = \sqrt{(s+24)(s)(9)(15)} = (s+24) \cdot \frac{27}{\sqrt{7}} \implies s = 81.$$

Thus, we compute the area as $(81 + 24) \cdot \frac{27}{\sqrt{7}} = \boxed{405\sqrt{7}}$.

Combinatorics

1. Twelve distinguishable socks, four of each of the colors red, blue, and green, are randomly placed into three bins with each bin receiving four socks. Compute the probability that each bin has exactly one pair of identically colored socks.

Proposed by Jacob Khohayting

Answer: $\frac{576}{1925}$

Suppose that two bins have the same pair. Then, the third bin must contain two pairs of the other two colors.

Assuming that the socks and bins are distinct, then there are $\binom{12}{4}\binom{8}{4}$ total ways to distribute the bins. Then, to distribute two socks of one type and one sock of the other two types in one bin, then there are 3! ways to determine which colored pair each bin receives. Then, for each color, there are $\binom{4}{2}\binom{2}{1}$ to distribute the socks. Hence, the answer is:

$$\frac{3! \cdot \left(\binom{4}{2}\binom{2}{1}\right)^3}{\binom{12}{4}\binom{8}{4}} = \boxed{\frac{576}{1925}}.$$

2. Ten apples of distinct sizes are randomly arranged in a circle. An alligator is first located at the smallest apple and eats it. Every second from there, the alligator looks at the two apples adjacent to him and eats the larger one, repeating until there is only one apple left, which he also eats. Compute the probability that the alligator eats the second largest apple before the largest apple.

Proposed by Jacob Khohayting

Answer: $\frac{35}{72}$

Label each apple with i where i is the rank of its size, with 1 being given to the smallest apple and 10 being given to the largest apple. We have two scenarios:

- If 9 and 10 are both next to 1, then 10 immediately is eaten. This occurs in 2 scenarios which 9 will lose in both.
- If 9 and 10 are not both next to 1, then we can swap 9 and 10. Then, the other value will be eaten first. So, there exists some symmetry. In the $9 \cdot 8 - 2$ scenarios, we find 9 will be eaten first in half of these.

Hence, the probability the alligator eats the second largest apple is

$$\frac{(9 \cdot 8 - 2)/2}{9 \cdot 8} = \boxed{\frac{35}{72}}.$$

3. Jason places each of his 100 marbles numbered 1 to 100 into one of two distinguishable jars so that each jar has at least one marble. Compute the number of ways he can do this so that the product of the numbers in each of the two jars are relatively prime to each other.

Proposed by Jacob Khohayting

Answer: 4094

First, place 1 in any jar. This has 2 ways. We now only consider prime numbers. After placing primes, then composites are immediately decided.

Consider all primes less than 50. Let p be a prime less than 50. After placing 2 in one of the jars, then $2p$ must also be in the same jar, and so p is also in the same jar. Placing the 2 comes with 2 ways.

Now, place the primes greater than 50 in any jar. All such primes include $\{53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$. There are 10 primes and so there are 2^{10} ways to place the marbles greater than 50.

So, there are 2^{12} ways to place marbles. We subtract 2 since the jars are non-empty yielding $2^{12} - 2 = \boxed{4094}$.

4. Ten identical pandas sit in a row of 14 chairs such that each chair is occupied by at most one panda. Each panda rolls either two chairs to the left or two chairs to the right. Compute the number of initial positions of the pandas such that it is possible for the pandas to occupy distinct chairs at the end.

Proposed by Jacob Khohayting

Answer: 596

Clearly, we can partition the fourteen chairs into the even positions and the odd positions.

Now, we consider a row of 7 chairs with indices 1 to 7. Notice, all pandas at odd indices will map to a elements at even indices, and vice versa. We don't care about even indices because we will always find a mapping. But, the odd indices matter since we can't place four pandas in three spots.

We will now perform simple casework on the number of pandas in odd/even indices:

- Suppose six pandas are in odd positions and four are in even positions.
  First consider the odd pandas, we require three of them to be in the sub-odd positions and three in the sub-even yielding $\binom{3}{3} \cdot \binom{4}{3} = 4$ ways.
  For the even pandas, we cannot sit the four in the sub-odd positions yielding $\binom{7}{4} - \binom{4}{4} \cdot \binom{3}{0} = 34$.
  Thus, $34 \cdot 4 = 136$.
- If five pandas are in odd positions and five are in even positions, then no configurations exist.
  Consider the odd pandas, then, we cannot sit four pandas in the sub-odd positions yielding $\binom{7}{5} - \binom{4}{4}\binom{3}{1} = 18$ ways.
  This is identical for the even pandas, so there are $18 \cdot 18 = 324$ ways.
- If four pandas are in odd positions and six are in even positions, then this case is identical to the second case above yielding 136 ways.

Hence, there are $136 + 324 + 136 = \boxed{596}$ total configurations.

5. In an archipelago of 5 islands, for each of the $\binom{5}{2}$ pairs of islands, Jacob has built a two-way bridge between them independently with probability $\frac{1}{2}$. He now collapses the maximum possible number of bridges so that any two islands that a hooligan could previously travel between by a sequence of bridges can still be traveled between. Compute the expected number of bridges remaining.

Proposed by Aaryan Vaishya

Answer: $\frac{1861}{512}$

Let $G \sim G(5, \frac{1}{2})$ be the random graph on the 5 islands. Collapsing the maximum number of bridges while preserving which vertices are mutually reachable means: inside each connected component of size k, keep only a spanning tree, so the number of bridges remaining equals $\sum(k-1) = 5 - c(G)$, where $c(G)$ is the number of connected components. Thus $\mathbb{E}[\text{bridges remaining}] = 5 - \mathbb{E}[c(G)]$. To compute $\mathbb{E}[c(G)]$, we first compute the connectivity probabilities $P_n = \Pr(G(n, \frac{1}{2}) \text{ is connected})$ via the recursion obtained by looking at the component of vertex 1: if that component has size k then we choose its other $k-1$ vertices, require the induced subgraph be connected, and require no edges from it to the complement. With $q = 1 - p = \frac{1}{2}$, this gives

$$P_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k \, q^{k(n-k)}, \qquad P_1 = 1.$$

Evaluating: $P_2 = 1 - q = \frac{1}{2}$, $P_3 = 1 - \left(\binom{2}{0}q^2 + \binom{2}{1}P_2 q^2\right) = \frac{1}{2}$, $P_4 = 1 - \left(\binom{3}{0}q^3 + \binom{3}{1}P_2 q^4 + \binom{3}{2}P_3 q^3\right) = \frac{19}{32}$, and

$$P_5 = 1 - \left(\binom{4}{0}q^4 + \binom{4}{1}P_2 q^6 + \binom{4}{2}P_3 q^6 + \binom{4}{3}P_4 q^4\right) = \frac{91}{128}.$$

For each k, let N_k be the number of connected components of size k in G. For a fixed subset S of k vertices, the event "S is a component" occurs iff the induced graph on S is connected (probability P_k) and there are no edges from S to its complement (probability $q^{k(5-k)}$), hence $\Pr(S \text{ is a component}) = P_k q^{k(5-k)}$. By linearity of expectation,

$$\mathbb{E}[N_k] = \binom{5}{k} P_k q^{k(5-k)}, \qquad \mathbb{E}[c(G)] = \sum_{k=1}^{5} \mathbb{E}[N_k] = \sum_{k=1}^{5} \binom{5}{k} P_k q^{k(5-k)}.$$

Substituting $q = \frac{1}{2}$ and the values above,

$$\mathbb{E}[c(G)] = 5 \cdot 1 \cdot \frac{1}{16} + 10 \cdot \frac{1}{2} \cdot \frac{1}{64} + 10 \cdot \frac{1}{2} \cdot \frac{1}{64} + 5 \cdot \frac{19}{32} \cdot \frac{1}{16} + 1 \cdot \frac{91}{128} = \frac{699}{512}.$$

Therefore

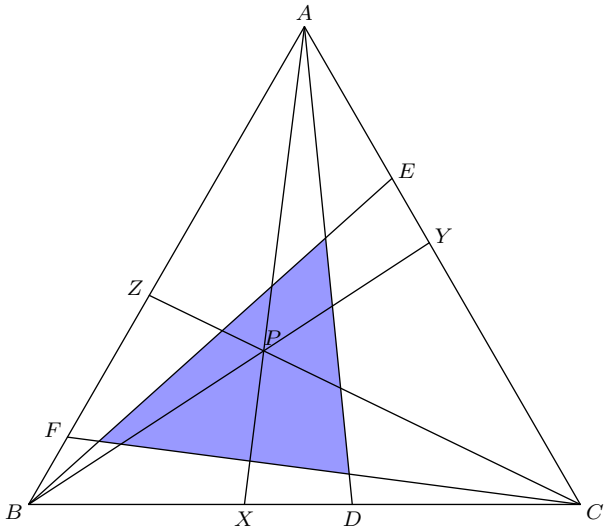
$$\mathbb{E}[\text{bridges remaining}] = 5 - \mathbb{E}[c(G)] = 5 - \frac{699}{512} = \boxed{\frac{1861}{512}}.$$

6. In triangle ABC with area 1, points D, E, F are chosen randomly and uniformly on sides BC, AC, AB respectively. The cevians AD, BE, CF form a triangle that does not share boundary points with any point on the perimeter of ABC. Compute the expected value of the area of this triangle.

Proposed by Aaryan Vaishya

Answer: $10 - \pi^2$

Note that the expected area of the inner triangle $\mathcal{T}$ is the probability that a randomly chosen point inside ABC lies inside $\mathcal{T}$. Consider a point P inside ABC with barycentric coordinates (x, y, z).



Let X, Y, Z be the foot of the cevians through P from A, B, C respectively. Then P lies inside the triangle formed by AD, BE, CF if and only if D, E, F lie on $\overline{BX}, \overline{CY}, \overline{AZ}$ respectively or on $\overline{CX}, \overline{AY}, \overline{BZ}$ respectively. The coordinates of X, Y, Z are given by

$$X = \left(0, \frac{y}{y+z}, \frac{z}{y+z}\right) \quad Y = \left(\frac{x}{z+x}, 0, \frac{z}{z+x}\right) \quad Z = \left(\frac{x}{x+y}, \frac{y}{x+y}, 0\right)$$

Swapping the order of integration to choose P first and then D, E, F, we find that the probability that P lies inside $\mathcal{T}$ is

$$\left(\prod_{\text{cyc}} \frac{y}{y+z}\right) + \left(\prod_{\text{cyc}} \frac{z}{y+z}\right) = \frac{2xyz}{(x+y)(y+z)(z+x)}$$

and that $\mathbb{E}[\mathcal{T}]$ is the average value of $\frac{2xyz}{(x+y)(y+z)(z+x)}$ over the triangle in $\mathbb{R}^3$ defined by $x, y, z \geq 0$ and $x + y + z = 1$. Writing $z = 1 - x - y$, this becomes $\frac{2xy(1-x-y)}{(x+y)(1-x)(1-y)}$, and projecting onto the plane $z = 0$, we get

$$\mathbb{E}[\text{Area}(\mathcal{T})] = 2 \iint_{\substack{x,y \geq 0 \\ x+y \leq 1}} \frac{2xy(1-x-y)}{(x+y)(1-x)(1-y)} \, dA$$

We evaluate

$$\begin{aligned}
2\iint_{\substack{x,y\geq 0\\ x+y\leq 1}} \frac{2xy(1-x-y)}{(x+y)(1-x)(1-y)}\,dA &= 2\int_0^1\int_0^{1-x} \frac{2xy(1-x-y)}{(x+y)(1-x)(1-y)}\,dy\,dx \\
&= \int_0^1 \frac{4x}{1-x}\int_0^{1-x} \frac{y(1-x-y)}{(x+y)(1-y)}\,dy\,dx \\
&= \int_0^1 \frac{4x}{1-x}\int_0^{1-x} \left(1-\left(\frac{x}{1+x}\right)\left(\frac{1}{x+y}+\frac{1}{1-y}\right)\right)\,dy\,dx \\
&= \int_0^1 \left(\frac{4x}{1-x}\right)\left((1-x)-\left(\frac{x}{1+x}\right)\left(\ln\left(\frac{1}{x}\right)-\ln\left(\frac{x}{1}\right)\right)\right)\,dx \\
&= \int_0^1 \left(4x+\frac{8x^2\ln(x)}{1-x^2}\right)\,dx \\
&= \left(\int_0^1 (4x-8\ln(x))\,dx\right)+\left(\int_0^1 \frac{8\ln(x)}{1-x^2}\,dx\right) \\
&= 10+\int_0^1 \frac{8\ln(x)}{1-x^2}\,dx \\
&\overset{x=e^{-t}}{=} 10-\int_0^\infty \frac{8te^{-t}}{1-e^{-2t}}\,dt \\
&= 10-8\int_0^\infty \left(te^{-t}+te^{-3t}+te^{-5t}+\cdots\right)\,dt \\
&= 10-8\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) \\
&= \boxed{10-\pi^2}
\end{aligned}$$