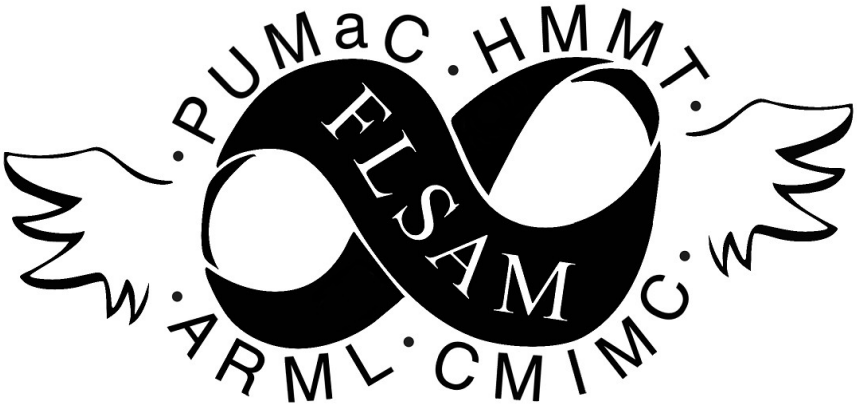
Florida Student Association of Mathematics



2025 Intro Meeting Test

September 2025

Welcome to the **2025 Intro Meeting Test**! This is the team round, consisting of 15 problems in the subjects of algebra, geometry, number theory, and combinatorics. Each problem is worth 1 point, and problems are not necessarily in order of increasing difficulty.

*You will have **60 minutes** to complete the test. Good luck, and have fun!*

Answer Key

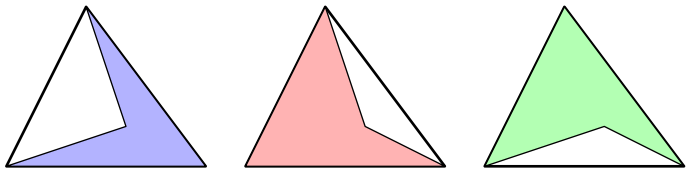
1. 4050
2. 82
3. $\frac{1}{9}$
4. 555
5. 207
6. $\frac{2-\sqrt{3}}{2}$
7. $\frac{\sqrt{13}}{13}$
8. 7776
9. $42\sqrt{11}$
10. $\frac{5}{32}$
11. 5050
12. 5047
13. $\frac{260}{11}$
14. 9
15. 45

1. Triangle $\mathcal{T}$ with area 2025 is drawn, and a point P is labeled in its interior. Jason, Daniel, and Jeffrey each draw a quadrilateral made up of vertices from $\mathcal{T}$ and P. To their surprise, they each drew a different figure! Find the sum of the areas of the quadrilaterals they drew.

Proposed by Jacob Khohayting

Answer: 4050

Notice, only three possible quadrilaterals can be created:



It suffices to compute the area of the three shaded regions. Notice, when the lengths from point P are drawn to each vertex of $\mathcal{T}$, then three triangles are formed. Each of the three triangles are appear twice among the shaded regions. Hence, the total area of the quadrilaterals is twice the area of the entire triangle. Thus, the answer is $2 \cdot 2025 = \boxed{4050}$.

2. Jason's favorite two-digit number is such that subtracting 2 or 4 from it results in a number that shares exactly one digit with the original, but subtracting 3 results in a number that shares no digits with the original. Find Jason's favorite number.

Proposed by Jacob Khohayting

Answer: 82

Notice, the units digit must be at most 2, since if it is greater than 2 then subtracting 3 yields the same tens digit.

Suppose the units digit is 2 first, say $\overline{n2}$. Then, subtracting 2 yields that the tens digit is the same. Subtracting 3 and 4 yields $\overline{(n-1)9}$ and $\overline{(n-1)8}$. Considering subtracting 4, then $n-1=2$ or $n=8$. If $n-1=2$, then the tens digit upon subtracting 3 is the same as the units digit of the original number. Thus, $n=8$. So, the number is 82 which works as subtracting 2, 3, and 4 yield 80, 79, and 78. This solution ensures uniqueness. Thus, the answer is $\boxed{82}$.

3. Ten people are lined up in a row, three of which are Aaryan, Ashvin, and Darsh. Given that Aaryan stands next to Ashvin, what is the probability Ashvin also stands next to Darsh?

Proposed by Jacob Khohayting

Answer: $\frac{1}{9}$

We consider two cases depending on the location of Ashvin:

- If Ashvin stands on the ends with probability $\frac{2}{18}$, then Darsh has no chance of standing next to Ashvin.
- If Ashvin doesn't stand on the ends with probability $\frac{16}{18}$, then Darsh has a probability of $\frac{1}{8}$ to sit next to Ashvin.

Thus, the probability Ashvin also stands next to Darsh is $\frac{2}{18} \cdot 0 + \frac{16}{18} \cdot \frac{1}{8} = \boxed{\frac{1}{9}}$.

4. Find the sum of the five smallest positive integers n such that $\gcd(n, 2100) = 15$.

Proposed by Hanfei He

Answer: 555

Notice, $15 \mid n$. Then, we can say $n = 15k$ for some positive integer k such that $\gcd(15k, 2100) = 15$. Equivalently, k satisfies $\gcd(k, 140) = 1$. Hence, k must be relatively prime to 2, 5, and 7. The least possible values of k are 1, 3, 9, 11, and 13. Thus, the sum of the smallest five positive integers are

$$15 \cdot (1 + 3 + 9 + 11 + 13) = \boxed{555}.$$

5. A set of primes is called *complete* if the digits of its elements are $1, 2, \ldots, 9$ in some order. For example, the set $\{7, 83, 421, 659\}$ is complete. Compute the smallest possible sum of elements of a complete set.

Proposed by Jingyi Li

Answer: 207

Notice, 4, 6, and 8 are all even numbers. If they were in the units digit of some number, then that value would be divisible by 2. Hence, these digits cannot be in the units digit.

So, if we can somehow place the remaining numbers in the units digit positions, then we would minimize the sum. In fact, we can with the construction $\{2, 3, 5, 41, 67, 89\}$. Thus, the minimum possible sum is:

$$2 + 3 + 5 + 41 + 67 + 89 = \boxed{207}.$$

6. Consider the function $f(x)$ satisfying

$$f(x + y) = f(x) + f(y)$$

for all real numbers x and y. Given $f(1) = 1$ and $f\left(\sqrt{2}\right) = \sqrt{3}$, compute $f\left(\frac{1}{2+\sqrt{2}}\right)$.

Proposed by Aaryan Vaishya

Answer: $\frac{2-\sqrt{3}}{2}$

Plugging in $x = y = \frac{\sqrt{2}}{2}$, then:

$$f\left(\sqrt{2}\right) = f\left(\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right) = 2 \cdot f\left(\frac{\sqrt{2}}{2}\right) \implies f\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{3}}{2}.$$

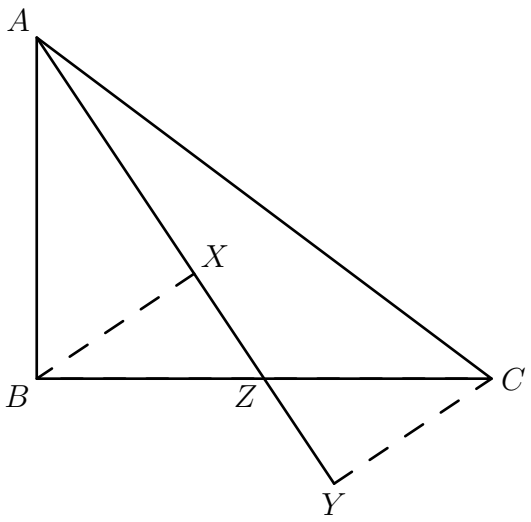
Now, we may compute the desired fraction:

$$f\left(\frac{1}{2+\sqrt{2}}\right) = f\left(\frac{2-\sqrt{2}}{2}\right) = f\left(1 - \frac{\sqrt{2}}{2}\right) = f(1) - f\left(\frac{\sqrt{2}}{2}\right) = 1 - \frac{\sqrt{3}}{2} = \boxed{\frac{2-\sqrt{3}}{2}}.$$

7. Triangle ABC has $\angle B = 90^\circ$ and centroid G. Points X and Y are on line AG such that $BX \perp AG$ and $CY \perp AG$. Suppose that $AB = 3$ and that $BC = 4$. Compute $AX - XY$.

Proposed by Luis Fonseca

Answer: $\frac{\sqrt{13}}{13}$



Suppose that $AX \cap BC = Z$. As G is the centroid, then AZ is a median. Therefore, Z is the midpoint of BC. Hence, $BZ = ZC = 2$. By Pythagorean theorem, then

$$AZ = \sqrt{AB^2 + BZ^2} = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

Note that $\triangle CYZ \sim \triangle ABZ$. Thus,

$$\frac{AZ}{BZ} = \frac{BZ}{XZ} \implies \sqrt{13} \cdot XZ = 2^2 \implies XZ = \frac{4}{\sqrt{13}} \implies AX = \frac{9}{\sqrt{13}}.$$

Also, $\triangle BZX \cong \triangle CZY$. Thus, $XZ = ZY$. Hence, $XY = XZ + ZY = \frac{8}{\sqrt{13}}$. Therefore,

$$AX - XY = \frac{9}{\sqrt{13}} - \frac{8}{\sqrt{13}} = \boxed{\frac{\sqrt{13}}{13}}.$$

8. Compute the number of five-digit numbers $\overline{ABCDE}$ such that none of

$$\overline{A}, \quad \overline{AB}, \quad \overline{ABC}, \quad \overline{ABCD}, \quad \overline{ABCDE}$$

are divisible by 3 or 10.

Proposed by Darsh Patel

Answer: 7776

As none of the five numbers are divisible by 10, then all digits are non-zero.

Consider A, B, C, D, and E modulo 3. The condition implies:

$$\begin{aligned} A &\not\equiv 0 \pmod 3, \\ A + B &\not\equiv 0 \pmod 3, \\ A + B + C &\not\equiv 0 \pmod 3, \\ A + B + C + D &\not\equiv 0 \pmod 3, \\ A + B + C + D + E &\not\equiv 0 \pmod 3. \end{aligned}$$

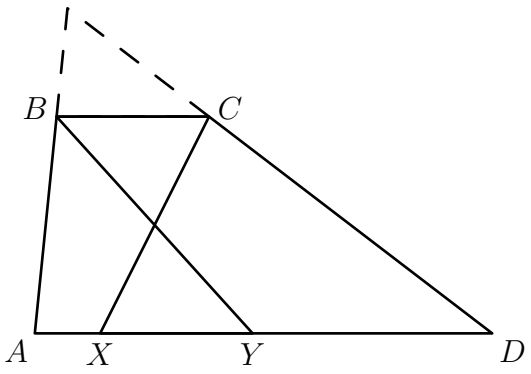
Firstly, A is either $1 \pmod 3$ or $2 \pmod 3$. Regardless of the choice for A, then B has two options as it cannot be the additive inverse of A. Similarly, C has two options as well since it cannot be the additive inverse of $A + B$, and so on. Hence, there are 2^5 ways to choose the moduli of the five digits modulo 3.

Now, for each variable, there are now 3 ways to choose the value. Therefore, the answer is $2^5 \cdot 3^5 = 6^5 = \boxed{7776}$.

9. Consider trapezoid $ABCD$ where $AD \parallel BC$ and $AD < BC$. The angle bisectors $\angle A$ and $\angle D$ intersect at P such that $\angle APB = \angle CPD = 90^\circ$. Suppose that $BA = 10$, $AD = 7$, and $DC = 18$. Find the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: $42\sqrt{11}$



Let the reflections of B and C over P be Y and X, respectively. As $\triangle BPC \cong \triangle YPX$, then

$$AD = AY + XD - XY = AB - BC + CD = 21.$$

Suppose $AB \cap DC = Q$. Then, $\triangle BQC \sim \triangle AQD$ with scale factor 3. Therefore:

$$\frac{QB}{QA} = \frac{BC}{AD} = \frac{QC}{QD} \implies \frac{QB}{QB+10} = \frac{1}{3} = \frac{QC}{QC+18}.$$

Therefore, $QB = 5$ and $QC = 9$. By Heron's, we may find $[BQC] = \frac{21\sqrt{11}}{4}$. Therefore:

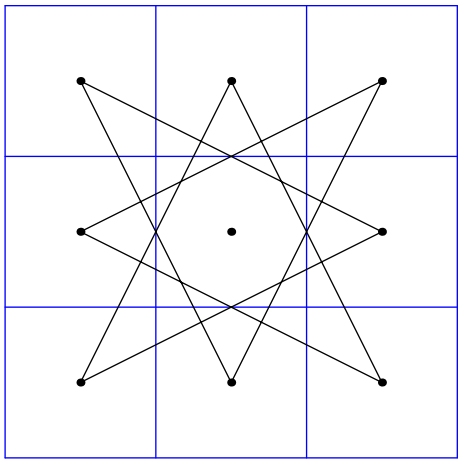
$$[ABCD] = [AQD] - [BQC] = \frac{21\sqrt{11}}{4} \cdot 9 - \frac{21\sqrt{11}}{4} = \boxed{42\sqrt{11}}.$$

10. Each tile of a 3×3 chessboard is colored either white or black at random. Find the probability that a knight can move from the bottom left tile to the top right tile without changing colors or a knight can move from the bottom right tile to the top left tile without changing colors, but not both.

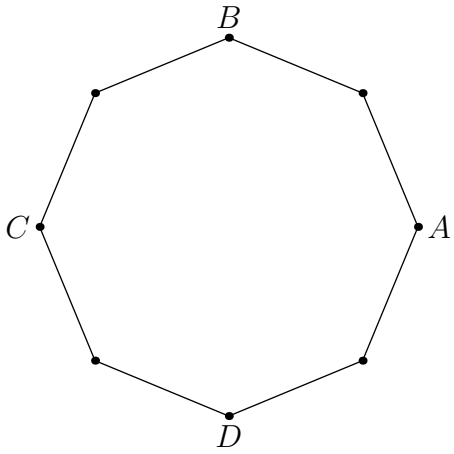
Proposed by Benjamin Jiang

Answer: $\frac{5}{32}$

We will compute the probability that the knight can get from the bottom left to the top right but not from the bottom right to the top left; the desired probability is twice this. Consider the graph on the tiles generated by knight moves:



Note that the knight can never get to the center tile. "Unwrapping" the graph, it becomes a cycle:



Here, A, B, C, and D represent the bottom left, bottom right, top right, and top left tiles respectively. Suppose the knight can get from A to C via the upper arc through B. This requires A, C, and the 3 tiles on the arc to all be the same color, which occurs with probability $\frac{1}{2^4}$. For the knight to not be able to get from B to D, we can check that 5 of the 8 ways to color the 3 tiles on the lower arc work. Adding the case where the knight can get from A to C via the lower arc, noting that these cases are disjoint, we get

$$2 \cdot \frac{1}{2^4} \cdot \frac{5}{8}$$

Then the desired probability is

$$2 \cdot 2 \cdot \frac{1}{2^4} \cdot \frac{5}{8} = \boxed{\frac{5}{32}}$$

11. A four-digit positive integer $\overline{ABCD}$ is *powerful* if A and C are nonzero and

$$\overline{ABCD} = \left(\overline{AB} + \overline{CD}\right)^2.$$

Find the sum of all powerful four-digit integers.

Proposed by George Paret

Answer: 5050

Let $X = \overline{AB}$, $Y = \overline{CD}$. Then

$$100X + Y = (X+Y)^2 \implies (X+Y)^2 = 100(X+Y) - 99Y.$$

Suppose $S = X + Y$. Modulo 99, then $99 \mid S(S-1)$. Therefore, taking modulo 9 and 11, then:

$$\begin{cases} S \equiv 0 \pmod 9 \\ S \equiv 1 \pmod 9 \end{cases} \qquad \begin{cases} S \equiv 0 \pmod{11} \\ S \equiv 1 \pmod{11} \end{cases}$$

By Chinese Remainder Theorem, then $S = \{0, 1, 45, 55, 99\}$. Crucially, 0 and 99 are both $0 \pmod{99}$. Clearly, $S \neq 0$ and $S \neq 1$. Now, we take each of these cases separately:

- If $S = 45$, then we obtain:

$$Y = \frac{100(X+Y) - (X+Y)^2}{99} = \frac{100 \cdot 45 - 45^2}{99} = 25.$$

  Hence, $X = 20$. Therefore, the desired number is 2025.

- If $S = 55$, then we obtain:

$$Y = \frac{100(X+Y) - (X+Y)^2}{99} = \frac{100 \cdot 55 - 55^2}{99} = 25.$$

  Hence, $X = 30$. Therefore, the desired number is 3025.

- If $S = 99$, then we obtain:

$$Y = \frac{100(X+Y) - (X+Y)^2}{99} = \frac{100 \cdot 99 - 99^2}{99} = 1.$$

  Hence, $X = 98$. Therefore, the desired number is 9801. Here $C = 0$, thus this is not a valid solution.

Thus, the desired answer is $2025 + 3025 = \boxed{5050}$.

12. Alice and Bob play a game on n points spaced equally on a circle. Alice goes first, and on each turn the player draws a chord on the circle between some pair of the points. The first player to draw a chord intersecting an existing chord loses. Compute the sum of all positive integers n between 3 and 100 such that Alice wins when the game is played optimally.

Proposed by Aaryan Vaishya

Answer: 5047

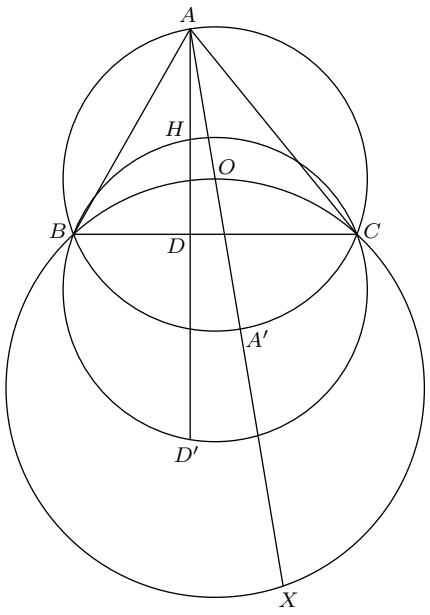
The game is forced to draw exactly the n sides of the polygon and $n-3$ diagonals in a triangulation, so there are always $2n-3$ moves and Alice always wins. The requested sum is then

$$3 + 4 + \cdots + 100 = \boxed{5047}.$$

13. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$, and circumcenter O. Suppose AO intersects the circumcircle of $\triangle BOC$ at $X \neq O$. Find the length of AX.

Proposed by Hanfei He

Answer: $\frac{260}{11}$



The main claim is that the image of X under $\sqrt{bc}$-inversion at A is the orthocenter of ABC. Let H be the orthocenter and DEF be the orthic triangle. Let A' be the A-antipode on (ABC) and D' be the reflection of A over D. First note that the image of (ABC) is BC, and so since AO and AH are isogonal it follows that $A'^* = D$. From this, taking the preimage and image of a homothety centered at A with scale factor 2 we obtain $O^* = D'$. But D' lies on (BHC) by the Orthocenter Reflection Lemma, so the image of (BOC) is (BHC). Now X^* is the second intersection of AD' with (BHC), which is H, as needed. Now since X and H are inverses, we have $AX \cdot AH = AB \cdot AC$.

We now also note that since $AEHF \stackrel{-}{\sim} ABA'C$, we have $AH = \frac{AE}{AB} \cdot AA' = 2R\cos(A)$. Then we can compute

$$AX = \frac{AB \cdot AC}{AH} = \frac{bc}{2R\cos(A)} = \frac{13 \cdot 15}{2 \cdot \frac{65}{8} \cdot \frac{33}{65}} = \boxed{\frac{260}{11}}.$$

Alternatively, since AH and AO are isogonal, then $\angle BAH = \angle XAC$. Also, BO and BH are isogonal. In conjunction with cyclic quadrilateral $BOCX$, then:

$$\angle AXC = \angle OXC = \angle OBC = \angle HBA.$$

Therefore, $\triangle ABH \sim \triangle AXC$. Thus, we obtain $\dfrac{AB}{AH} = \dfrac{AX}{AC}$. We can now proceed as before.

14. Let a, b, and c be positive real numbers such that

$$a^2 = bc - 8,$$
$$b^2 = ca + 1,$$
$$c^2 = ab + 10.$$

Find $a + b + c$.

Proposed by Joshua Khohayting

Answer: 9

Subtracting pairs:

$$(b^2 - ca) - (a^2 - bc) = (b - a)(a + b + c) = 9,$$
$$(c^2 - ab) - (b^2 - ca) = (c - b)(a + b + c) = 9,$$
$$(a^2 - bc) - (c^2 - ab) = (a - c)(a + b + c) = -18.$$

Let $b - a = N$, $c - b = N$, and $a - c = -2N$ where $a + b + c = \frac{9}{N}$. But,

$$\frac{(b-a)^2 + (c-b)^2 + (a-c)^2}{2} = a^2 + b^2 + c^2 - ab - bc - ca.$$

Thus, $3N^2 = 3$. Hence, $N^2 = 1$. Therefore, $a + b + c = \boxed{9}$. Remarkably, the solution is $(2, 3, 4)$.

15. Triangle $\triangle$ in $\mathbb{R}^3$ has projections $\triangle_{xy}$, $\triangle_{yz}$, and $\triangle_{zx}$ onto the xy, yz, and zx planes, respectively. Given that the areas of $\triangle_{xy}$, $\triangle_{yz}$, and $\triangle_{zx}$ are 5, 8, and 44 respectively, find the area of $\triangle$.

Proposed by Roger He

Answer: 45

Suppose $\triangle$ is made up of vectors (a_1, a_2, a_3) and (b_1, b_2, b_3). Therefore:

$$5 = [\triangle_{xy}] = \tfrac{1}{2}|(a_1, a_2, 0) \times (b_1, b_2, 0)| = \tfrac{1}{2}|a_1 b_2 - a_2 b_1|,$$
$$8 = [\triangle_{yz}] = \tfrac{1}{2}|(0, a_2, a_3) \times (0, b_2, b_3)| = \tfrac{1}{2}|a_2 b_3 - a_3 b_2|,$$
$$44 = [\triangle_{zx}] = \tfrac{1}{2}|(a_1, 0, a_3) \times (b_1, 0, b_3)| = \tfrac{1}{2}|a_3 b_1 - a_1 b_3|.$$

Notice, we finish:

$$\begin{aligned} 5^2 + 8^2 + 44^2 &= \tfrac{1}{4}\left[(a_1 b_2 - a_2 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2\right] \\ &= \left(\tfrac{1}{2}|(a_1, a_2, a_3) \times (b_1, b_2, b_3)|\right)^2 \\ &= [\triangle]^2. \end{aligned}$$

Therefore, $[\triangle] = \sqrt{5^2 + 8^2 + 44^2} = \sqrt{2025} = \boxed{45}$.