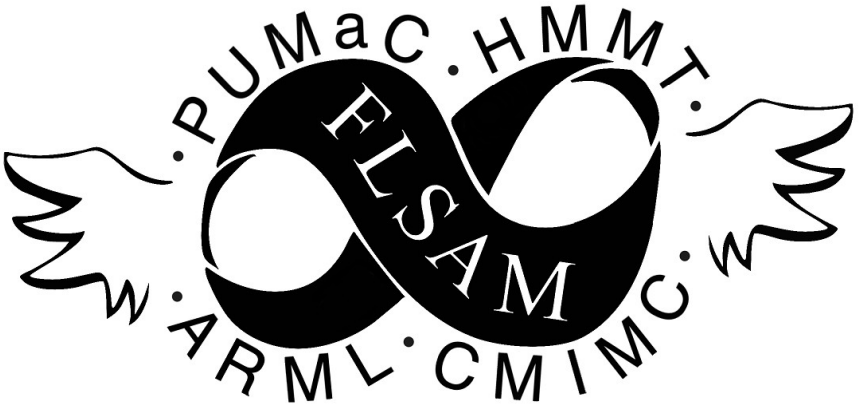
Florida Student Association of Mathematics



2025 PUMAC Tryout

October 25-26, 2025

Solutions

Algebra Answers

1. 17
2. $\frac{59}{3}$
3. 404
4. 137

Number Theory Answers

1. 22110
2. 1449
3. 396
4. $\frac{1012}{1013}$

Geometry Answers

1. 240
2. $27\sqrt{2}$
3. $5184\sqrt{2}$
4. $84\sqrt{143}$

Combinatorics Answers

1. $\frac{152}{243}$
2. $\frac{4}{21}$
3. 1581
4. 960000

Algebra

1. Distinct reals a, b, c are the roots of the cubic $x^3 - 10x^2 + kx - 2$, for some real k. Given that $ax^2 + bx + c = 0$ has a unique real solution, compute k.

Proposed by Darsh Patel

Answer: 17

Taking the discriminant, we must have $b^2 = 4ac$. Now by Vieta formulas, we have

$$a + b + c = 10$$
$$ab + ac + bc = k$$
$$abc = 2$$

Then $b^3 = 4abc = 8$ and so $b = 2$. We now have

$$k = ab + ac + bc = b(a + c) + \frac{abc}{b} = 2(10 - 2) + \frac{2}{2} = \boxed{17}$$

2. The graphs of $y = x^3 - 5x^2 + 7x + 4$ and $y = 2x^2 - 6$ intersect at A, B, and C. Find the sum of the coordinates of the centroid of triangle ABC.

Proposed by Jacob Khohayting

Answer: $\frac{59}{3}$

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the points of intersection. Then, x_1, x_2, and x_3 must satisfy

$$x^3 - 5x^2 + 7x + 4 = 2x^2 - 6 \iff x^3 - 7x^2 + 7x + 10 = 0.$$

Thus, by Vieta's, $x_1 + x_2 + x_3 = 7$ and $x_1x_2 + x_2x_3 + x_3x_1 = 7$. Therefore,

$$\begin{aligned}\sum_{\text{cyc}} y_i &= \sum_{\text{cyc}} \left(2x_i^2 - 6\right) \\ &= 2 \cdot \sum_{\text{cyc}} x_i^2 - 18 \\ &= 2 \cdot \Big((x_1 + x_2 + x_3)^2 - 2 \cdot (x_1x_2 + x_2x_3 + x_3x_1)\Big) - 18 \\ &= 52.\end{aligned}$$

Hence, the sum of the coordinates of the centroid is equivalent to

$$\frac{1}{3} \cdot \left(\sum_{\text{cyc}} x_i + \sum_{\text{cyc}} y_i\right) = \boxed{\frac{59}{3}}$$

3. For each nonnegative integer k, $P_k(x)$ is a real polynomial of degree k with leading coefficient $\frac{k+1}{k!}$. For each k, we have that

$$P_k(k) = P_{k+1}(k) = P_{k+2}(k) = \cdots$$

Given that $P_{100}(100)$ is an integer, compute the number of positive integer divisors of $P_{100}(100)$.

Proposed by Darsh Patel

Answer: 404

Let $c_k = k+1$, so that $P_k(x)$ has leading coefficient $\frac{c_k}{k!}$. Let $a_k = P_k(k)$. Then we are given that for each k,

$$(a_0, a_1, \ldots, a_k) = (P_k(0), P_k(1), \ldots, P_k(k))$$

Note that $P_k(x) - P_{k-1}(x)$ has roots at $0, 1, \ldots, k-1$. This implies that

$$P_k(x) = P_{k-1}(x) + \frac{c_k}{k!}x(x-1)\ldots(x-k+1) = P_{k-1}(x) + c_k\binom{x}{k}$$

Thus,

$$P_k(x) = c_k\binom{x}{k} + c_{k-1}\binom{x}{k-1} + \cdots + c_0\binom{x}{0}$$

We now compute

$$\begin{aligned}
&P_{100}(100)\\
=&101\binom{100}{100} + 100\binom{100}{99} + \cdots + 1\binom{100}{0}\\
=&\left(\binom{100}{100} + \binom{100}{99} + \cdots + \binom{100}{0}\right) + 100\left(\binom{99}{99} + \binom{99}{98} + \cdots + \binom{99}{0}\right)\\
=&2^{100} + 100(2^{99})\\
=&2^{100}\cdot 3\cdot 17
\end{aligned}$$

Then the desired value is $\boxed{404}$.

4. The values $a_1, a_2, \ldots, a_{20}$ are complex numbers. A sequence $\{s_n\}_{n\geq 1}$ of complex numbers is called *princely* if

$$s_n = a_1 s_{n-1} + a_2 s_{n-2} + \cdots + a_{20} s_{n-20}$$

for all $n > 20$. Princely sequences $\{s_{1,n}\}_{n\geq 1}, \{s_{2,n}\}_{n\geq 1}, \ldots, \{s_{20,n}\}_{n\geq 1}$ are such that

- $\left|\frac{s_{1,n}}{s_{2,n}}\right|, \left|\frac{s_{2,n}}{s_{3,n}}\right|, \ldots, \left|\frac{s_{19,n}}{s_{20,n}}\right| > n$ for all positive integers n, and
- $n^{k-\frac{1}{2}} < |s_{1,n}s_{2,n}\ldots s_{20,n}| < n^{k+\frac{1}{2}}$ for infinitely many positive integers n.

Find the 6th largest possible integer value of k.

Proposed by Darsh Patel

Answer: 137

First assume $a_{20} \neq 0$. We have a linear recurrence with characteristic polynomial

$$P(x) = x^{20} - a_1x^{19} - a_2x^{18} - \cdots - a_{20}$$

The following fact is well-known: if $P(x) = (x-r_1)^{d_1}(x-r_2)^{d_2}\ldots(x-r_m)^{d_m}$ for distinct $r_1, r_2, \ldots, r_m$ and $d_1, d_2, \ldots, d_m \geq 1$, then the solutions $\{s_n\}_{n\geq 1}$ to the recurrence (i.e. princely sequences) are the linear combinations of

$$\begin{gathered}
r_1^n, nr_1^n, \ldots n^{d_1-1}r_1^n\\
r_2^n, nr_2^n, \ldots, n^{d_2-1}r_2^n\\
\vdots\\
r_m^n, nr_m^n, \ldots, n^{d_m-1}r_m^n
\end{gathered}$$

Now clearly for the second condition to hold, our 20 princely sequences must be nonzero. Define the *growth rate* of a nonzero princely sequence $\{s_n\}$ to be the sequence $n^d a^n$, where $n^d r^n$ is a term appearing in the formula

for the princely sequence with nonzero coefficient and with $a = |r|$ maximal and then d maximal. We will first show that $0 < \limsup_{n\to\infty} \frac{s_n}{n^d a^n} < \infty$. Suppose the sum of terms of the form $n^d r^n$ in the formula of $\{s_n\}$ having $|r| = a$ is

$$c_1 n^d r_{i_1}^n + c_2 n^d r_{i_2}^n + \cdots + c_\ell n^d r_{i_\ell}^n$$

with $c_1, c_2, \ldots, c_\ell$ not all 0. Then

$$\frac{s_n}{n^d a^n} = c_1 \left(\frac{r_{i_1}}{a}\right)^n + c_2 \left(\frac{r_{i_2}}{a}\right)^n + \cdots + c_\ell \left(\frac{r_{i_\ell}}{a}\right)^n + O\left(\frac{1}{n}\right)$$

Now since $c_1, c_2, \ldots, c_\ell$ are not all 0,

$$\limsup_{n\to\infty} \frac{|s_n|}{n^d a^n} = \limsup_{n\to\infty} \left| c_1 \left(\frac{r_{i_1}}{a}\right)^n + c_2 \left(\frac{r_{i_2}}{a}\right)^n + \cdots + c_\ell \left(\frac{r_{i_\ell}}{a}\right)^n \right| > 0$$

We also have

$$\limsup_{n\to\infty} \frac{|s_n|}{n^d a^n} \le |c_1| + |c_2| + \cdots + |c_n| < \infty$$

by Triangle Inequality.

The main claim is now that if $\{s_{1,n}\}, \{s_{2,n}\}$ have growth rates $n^d a^n, n^e b^n$ respectively, then $n^e b^n = o(n^d a^n)$, and similarly for all other consecutive pairs. This follows our previous bound about the growth rates of princely sequences and fact that $\left|\frac{s_{1,n}}{s_{2,n}}\right| > n$ for all n.

Now, there are at most 20 distinct growth rates, equality holding when $|r_1|, |r_2|, \ldots, |r_m|$ are distinct. Thus, these values are distinct, and assuming WLOG $|r_1| > |r_2| > \cdots > |r_m|$, we have that $\{s_{1,n}\}, \{s_{2,n}\}, \ldots, \{s_{20,n}\}$ have growth rates

$$\begin{gathered} n^{d_1-1}|r_1|^n, n^{d_1-2}|r_1|^n, \ldots, |r_1|^n, \\ n^{d_2-1}|r_2|^n, n^{d_2-2}|r_2|^n, \ldots, |r_2|^n, \\ \vdots \\ n^{d_m-1}|r_m|^n, n^{d_m-2}|r_m|^n, \ldots, |r_m|^n \end{gathered}$$

respectively. This is also enough to imply that in fact $\lim_{n\to\infty} \frac{s_n}{n^d a^n}$ exists and is in $(0, \infty)$ for any princely sequence $\{s_n\}$ with growth rate $n^d a^n$. At this point, we see that if $a_{20} = 0$, then we would instead have a linear recurrence of degree at most 19 and so have at most 19 distinct growth rates. This is impossible, so we can safely continue with the assumption $a_{20} \neq 0$. Multiplying the growth rates, we now have

$$|s_{1,n} s_{2,n} \ldots s_{20,n}| = \Theta\left(n^{\binom{d_1}{2}+\binom{d_2}{2}+\cdots+\binom{d_m}{2}} \left| r_1^{d_1} r_2^{d_2} \ldots r_m^{d_m} \right|^n\right)$$

This implies $\left| r_1^{d_1} r_2^{d_2} \ldots r_m^{d_m} \right| = 1$ and $k = \binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_m}{2}$.

Thus, we wish to compute the 6th largest possible value of $\binom{d_1}{2} + \binom{d_2}{2} + \cdots + \binom{d_{20}}{2}$ over nonnegative integers

$d_1, d_2, \ldots, d_{20}$ with sum 20. The 8 largest such values are

$$\begin{aligned}
&\binom{20}{2} \\
>&\binom{19}{2} + \binom{1}{2} \\
>&\binom{18}{2} + \binom{2}{2} \\
>&\binom{18}{2} + \binom{1}{2} + \binom{1}{2} \\
>&\binom{17}{2} + \binom{3}{2} \\
>&\binom{17}{2} + \binom{2}{2} + \binom{1}{2} \\
>&\binom{17}{2} + \binom{1}{2} + \binom{1}{2} + \binom{1}{2} \\
>&\binom{16}{2} + \binom{4}{2}
\end{aligned}$$

where we note that $\binom{16}{2} + \binom{4}{2}$ is indeed larger than all other possible values as a consequence of Karamata Inequality, since $\binom{x}{2}$ is convex as a function in x. Thus the desired value is $\binom{17}{2} + \binom{2}{2} + \binom{1}{2} = \boxed{137}$.

Number Theory

1. Find the smallest positive integer n such that the number of positive integers k at most n with $\gcd(k, n) = 1$ is exactly $\frac{16}{67}n$.

Proposed by Darsh Patel

Answer: 22110

We are given

$$\frac{\varphi(n)}{n} = \frac{16}{67}$$

If $p_1 < p_2 < \cdots < p_k$ are the prime divisors of n, then we have

$$\left(1 - \frac{1}{p_1}\right) \ldots \left(1 - \frac{1}{p_k}\right) = \frac{16}{67}$$

Now the key observation is that in general if we replace $\frac{16}{67}$ with some reduced fraction $\frac{a}{b}$, then p_k is the largest prime divisor of b. We would then obtain

$$\left(1 - \frac{1}{p_1}\right) \ldots \left(1 - \frac{1}{p_{k-1}}\right) = \frac{a}{b} \cdot \frac{p_k}{p_k - 1}$$

We could then perform the same procedure on $\frac{a}{b} \cdot \frac{p_k}{p_k - 1}$, and so on until we eventually get 1. Doing this, we

get

$$\begin{aligned}
&\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_k}\right)=\frac{16}{67} \\
\implies &\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_{k-1}}\right)=\frac{8}{33} && p_k=67 \\
\implies &\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_{k-2}}\right)=\frac{4}{15} && p_{k-1}=11 \\
\implies &\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_{k-3}}\right)=\frac{1}{3} && p_{k-2}=5 \\
\implies &\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_{k-4}}\right)=\frac{1}{2} && p_{k-3}=3 \\
\implies &\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_{k-5}}\right)=1 && p_{k-4}=2
\end{aligned}$$

Then $k=5$ and the smallest possible n is

$$2\cdot 3\cdot 5\cdot 11\cdot 67=\boxed{22110}$$

2. Find the smallest positive integer n such that

$$3\left\lfloor\frac{n}{3}\right\rfloor-4\left\lfloor\frac{n}{4}\right\rfloor=7\left\lfloor\frac{n}{7}\right\rfloor-8\left\lfloor\frac{n}{8}\right\rfloor=11\left\lfloor\frac{n}{11}\right\rfloor-12\left\lfloor\frac{n}{12}\right\rfloor=1.$$

Proposed by Jacob Khohayting

Answer: 1449

Let $r_m(n)$ denote the remainder when n is divided by m. We have

$$3\left\lfloor\frac{n}{3}\right\rfloor-4\left\lfloor\frac{n}{4}\right\rfloor=1 \implies \left(n-4\left\lfloor\frac{n}{4}\right\rfloor\right)-\left(n-3\left\lfloor\frac{n}{3}\right\rfloor\right)=1 \implies r_4(n)-r_3(n)=1$$

Similarly, $r_8(n)-r_7(n)=r_{12}(n)-r_{11}(n)=1$. Let $k=r_3(n)$. Then $r_4(n)=k+1$. By Chinese Remainder Theorem, $r_{12}(n)=k+9$, and so $r_{11}(n)=k+8$. Now, we have that $r_8(n)$ is one of $k+1, k+5$. If $r_8(n)=k+5$, then $r_7(n)=k+4$ and

$$n\equiv k-3 \pmod{1848}$$

where $1848=\operatorname{lcm}(3,4,7,8,11,12)$. This yields a smallest possible n of $1848-3$, which is too large. Otherwise, $r_8(n)=k+1$, $r_7(n)=k$, and by Chinese Remainder Theorem

$$n\equiv k+1449 \pmod{1848}$$

This yields a smallest possible n of $\boxed{1449}$, achieved at $k=0$.

3. A sequence $\{a_n\}_{n\geq 1}$ of positive integers is defined recursively by $a_1=k$ and

$$\begin{cases} a_{n+1}=\frac{1}{2}\cdot a_n & \text{if } a_n\equiv 0 \pmod 2 \\ a_{n+1}=a_n+77 & \text{if } a_n\equiv 1 \pmod 2 \end{cases}$$

for all n. Find the sum of all values of k for which $a_6=a_1$.

Proposed by Jacob Khohayting

Answer: 396

Note that $a_1 \equiv a_6 \pmod{77}$. Note the second operation does not change $a_i \pmod{77}$. Now, we will compute the number of times the first operation is applied. To do this, consider the sequence of remainders modulo 7 and 11 obtained by only applying the first condition.

$$A \longrightarrow 4A \longrightarrow 2A \longrightarrow A \pmod 7$$
$$A \longrightarrow 6A \longrightarrow 3A \longrightarrow 7A \longrightarrow 9A \longrightarrow 10A \longrightarrow 5A \longrightarrow 8A \longrightarrow 4A \longrightarrow 2A \longrightarrow A \pmod{11}.$$

Thus, we require $a_1 \equiv 0 \pmod{11}$. In addition, either $a_1 \equiv 0 \pmod 7$ or we apply the first operation three times and the second operation twice.

If $a_1 \equiv 0 \pmod{77}$, one can scale down by 77 to remove this condition. So, define $b_i = \frac{a_i}{77}$. Then, $b_{n+1} = \frac{1}{2} \cdot b_n$ if $b_n \equiv 0 \pmod 2$ and $b_{n+1} = b_n + 1$ if $b_n \equiv 1 \pmod 2$. Notice, if we apply the second operation (2), we must apply the first operation (1) immediately. Consider casework on parity of the first number.

- If $b_1 \equiv 0 \pmod 2$, then we must apply the first operation (1). From here, we have a few cases to consider:

$$(1)(1)(2)(1)(2) \implies X \to \tfrac{X}{2} \to \tfrac{X}{4} \to \tfrac{X}{4}+1 \to \tfrac{X}{8}+\tfrac{1}{2} \to \tfrac{X}{8}+\tfrac{3}{2} \implies X = \tfrac{12}{7}$$
$$(1)(2)(1)(1)(2) \implies X \to \tfrac{X}{2} \to \tfrac{X}{2}+1 \to \tfrac{X}{4}+\tfrac{1}{2} \to \tfrac{X}{8}+\tfrac{1}{4} \to \tfrac{X}{8}+\tfrac{5}{4} \implies X = \tfrac{10}{7}.$$
$$(1)(2)(1)(2)(1) \implies X \to \tfrac{X}{2} \to \tfrac{X}{2}+1 \to \tfrac{X}{4}+\tfrac{1}{2} \to \tfrac{X}{4}+\tfrac{3}{2} \to \tfrac{X}{8}+\tfrac{3}{4} \implies X = \tfrac{6}{7}.$$

  Notice, $(2)(1)(1)(1)(2)$ is impossible as by applying the $(1)(2)$ at the end, then $b_6 \equiv 0 \pmod 2$, a contradiction. None of these cases are satisfactory as X is not an integer in any case.

- If $b_1 \equiv 1 \pmod 2$, then we must apply the first operation (2). From here, we have a few cases to consider:

$$(2)(1)(2)(1)(1) \implies X \to X+1 \to \tfrac{X}{2}+\tfrac{1}{2} \to \tfrac{X}{2}+\tfrac{3}{2} \to \tfrac{X}{4}+\tfrac{3}{4} \to \tfrac{X}{8}+\tfrac{3}{8} \implies X = \tfrac{3}{7}.$$
$$(2)(1)(1)(2)(1) \implies X \to X+1 \to \tfrac{X}{2}+\tfrac{1}{2} \to \tfrac{X}{4}+\tfrac{1}{4} \to \tfrac{X}{4}+\tfrac{5}{4} \to \tfrac{X}{8}+\tfrac{5}{8} \implies X = \tfrac{5}{7}.$$

  None of these cases are satisfactory as X is not an integer in any case.

Suppose we were to apply the first operation three times and the second operation twice. Note that these cases are also just scaling the cases from before back up by 77 as they have three (1)'s and two (2)'s, but we must check to ensure X is divisible by 11 and satisfies all other conditions.

- If $a_1 \equiv 0 \pmod 2$, then we require the following cases:

$$(1)(1)(2)(1)(2) : 132 \to 66 \to 33 \to 110 \to 55 \to 132$$
$$(1)(2)(1)(1)(2) : 110 \to 55 \to 132 \to 66 \to 33 \to 110$$
$$(1)(2)(1)(2)(1) : 66 \to 33 \to 110 \to 55 \to 132 \to 66$$

  It is easy to confirm all these work.

- If $a_1 \equiv 1 \pmod 2$, then we require the following cases:

$$(2)(1)(2)(1)(1) : 33 \to 110 \to 55 \to 132 \to 66 \to 33$$
$$(2)(1)(1)(2)(1) : 55 \to 132 \to 66 \to 33 \to 110 \to 55$$

  Notice, $(2)(1)(1)(1)(2)$ is impossible as by applying the $(1)(2)$ at the end, then $a_6 \equiv 0 \pmod 2$, a contradiction. It is easy to confirm all these work.

Thus, the sum is $132 + 110 + 66 + 33 + 55 = \boxed{396}$.

4. Aaryan calculates the product $2025 \cdot 2026$ by starting with $S = 0$ and adding 2025 to S until he gets $S = 2025 \cdot 2026$. Compute the average number of regroups/carries he does per addition over all 2026 additions.

[For example, the addition $9843 + 417$ involves 3 regroups/carries, occurring at the ones, hundreds, and thousands digits.]

Proposed by Darsh Patel

Answer: $\frac{1012}{1013}$

Consider any single addition. Imagine Aaryan first performs the addition without carries by allowing each digit to exceed 9, and then performs the carries. The main idea is to look at digital sum; each carry decreases a digit by 10 and increases the next by 1, decreasing the digital sum by 9 in total. Thus, the total number of regroups is just the difference between the starting digital sum and ending digital sum, divided by 9. We compute

$$\frac{2026s_{10}(2025) - s_{10}(2025 \cdot 2026)}{9} = \frac{2026 \cdot 9 - 18}{9} = 2024$$

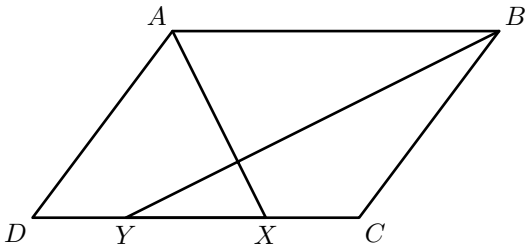
Then the desired value is $\frac{2024}{2026} = \boxed{\frac{1012}{1013}}$.

Geometry

1. In parallelogram $ABCD$, the angle bisectors of $\angle A$ and $\angle B$ intersect segment CD and divide the interior of the parallelogram into four regions: two triangles and two quadrilaterals. Given that the triangles have areas 27 and 75, find the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: 240



Let P be the intersection of the angle bisectors of $\angle A$ and $\angle B$. Let X and Y be the intersection of angle bisectors of $\angle A$ and $\angle B$ with CD, respectively.

Easily, we find:

$$[ABCD] = 2 \cdot [ABC] = 2 \cdot \left(\frac{AB \cdot d(C, AB)}{2}\right) = AB \cdot (d(P, CD) + d(P, AB))$$

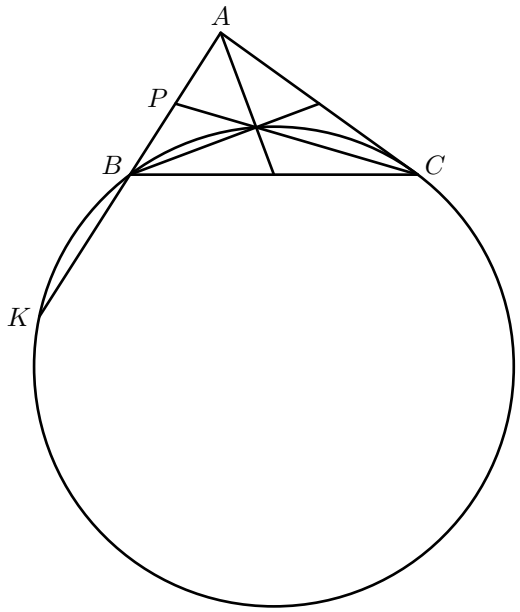
But, $\triangle APB \sim \triangle YPX$ with ratio $\frac{5}{3}$. Thus, $d(P, AB) = \frac{5}{3} \cdot d(P, CD)$.

$$[ABCD] = AB \cdot \left(d(P, AB) \cdot \tfrac{8}{5}\right) = \left(\tfrac{1}{2} \cdot AB \cdot d(P, AB)\right) \cdot \tfrac{16}{5} = [APB] \cdot \tfrac{16}{5} = 75 \cdot \tfrac{16}{5} = \boxed{240}.$$

2. Let ABC be a triangle with centroid G. Suppose the reflection of A across B lies on the circumcircle of BGC. Given that $GB = 3$ and $GC = 9$, find the area of ABC.

Proposed by Jacob Khohayting

Answer: $27\sqrt{2}$



Let $P = CG \cap AB$; that is, P is the midpoint of AB. By Power of Point, we find:

$$PB \cdot PK = PG \cdot PC \implies PB \cdot (3 \cdot PB) = PG \cdot (3 \cdot PG) \implies PB = PG.$$

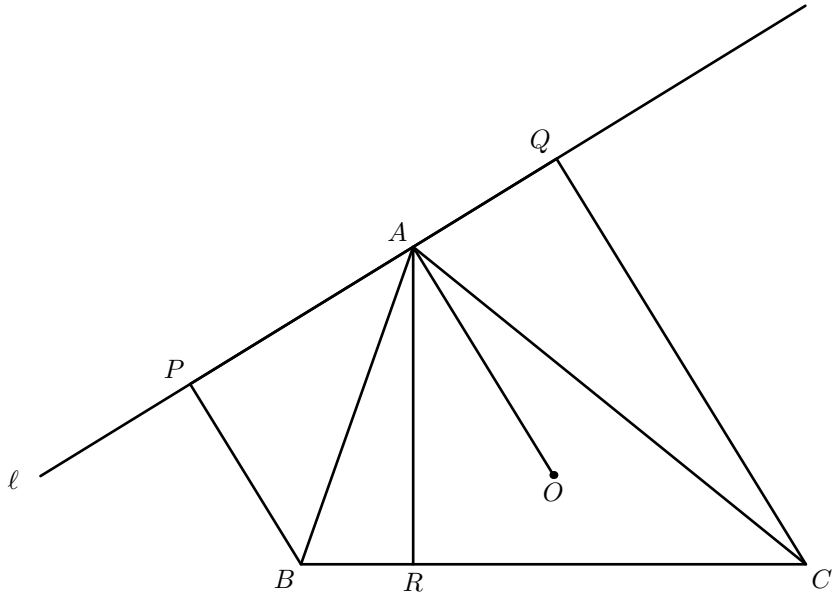
Thus, P is the center of (ABG). Thus, $\angle ABG = 90^\circ$ and $AB = 9$. By Pythagorean Theorem, then we find $AG = 6\sqrt{2}$. Now, we may now compute the area of $\triangle ABC$:

$$[ABC] = 3 \cdot [ABG] = 3 \cdot \frac{AG \cdot BG}{2} = 3 \cdot \frac{6\sqrt{2} \cdot 3}{2} = \boxed{27\sqrt{2}}.$$

3. Acute triangle ABC has circumcenter O and circumradius 81. Let line ℓ be the tangent to the circumcircle of ABC at A. Given that the distances from B and C to ℓ are 64 and 144 respectively, find the area of ABC.

Proposed by Jacob Khohayting

Answer: $5184\sqrt{2}$



Let P and Q be the feet of the perpendiculars from B and C to ℓ, respectively. In addition, let R be the foot of the perpendicular from A to BC.

Notice, $\triangle ABP \sim \triangle CAR$ and $\triangle BAR \sim \triangle QCA$. Therefore:

$$\frac{AB}{PB} = \frac{AC}{AR} \implies \frac{AB}{64} = \frac{AC}{AR} \implies 64 = \frac{AB \cdot AR}{AC}$$
$$\frac{AC}{QC} = \frac{AB}{AR} \implies \frac{AC}{144} = \frac{AB}{AR} \implies 144 = \frac{AC \cdot AR}{AB}.$$

Hence, we may find:

$$\frac{144}{64} = \frac{(AC \cdot AR)/AB}{(AB \cdot AR)/AC} = \left(\frac{AC}{AB}\right)^2 \implies \frac{AC}{AB} = \frac{3}{2}.$$

Thus, $AR = 96$. Let $AB = 2a$ and $AC = 3a$. Also, let $BC = s$. Notice, $OA = d(O, \ell) = 81$. Therefore, using the circumradius formula, we find:

$$[ABC] = \frac{AB \cdot BC \cdot CA}{4 \cdot OA} = \frac{2a \cdot BC \cdot 3a}{4 \cdot 81}.$$

But, $[ABC] = \frac{1}{2} \cdot BC \cdot AR$. Thus:

$$\frac{1}{2} \cdot BC \cdot 96 = \frac{2a \cdot BC \cdot 3a}{324} \implies \frac{96 \cdot BC}{2} = \frac{6a^2 \cdot BC}{324} \implies a = 36\sqrt{2}.$$

Thus, we find:

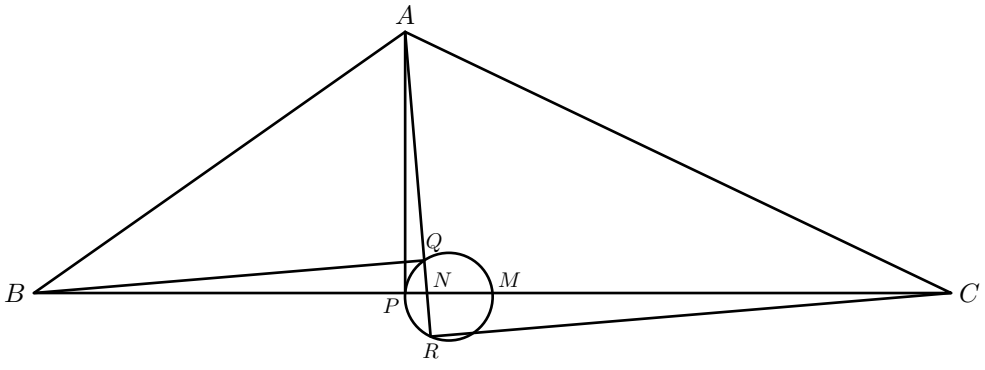
$$BC = \sqrt{(AB)^2 - (AR)^2} + \sqrt{(AC)^2 - (AR)^2} = \sqrt{(72\sqrt{2})^2 - 96^2} + \sqrt{(108\sqrt{2})^2 - 96^2} = 108\sqrt{2}.$$

But, as $AR = 96$, then $[ABC] = \boxed{5184\sqrt{2}}$.

4. Let ABC be an acute triangle and ℓ be the angle bisector of $\angle A$. Let N be the intersection of ℓ with BC, and P, Q, and R be the feet of the perpendiculars from A to BC, B to ℓ, and C to ℓ respectively. Given that $PN = 2$, $QN = 3$, and $RN = 4$, find the area of ABC.

Proposed by Jacob Khohayting

Answer: $84\sqrt{143}$



Let M be the midpoint of BC. Notice, the reflections of B over Q and M lie on AC. So, M and Q both lie on the B-midline. Hence, $QM \parallel AC$. Similarly, $RM \parallel AB$. Therefore,

$$\measuredangle QPM = \measuredangle QPB = \measuredangle RAB = \measuredangle QRM.$$

Hence, $PQMR$ is cyclic. Then, by Power of Point at N, then $NM = 6$. Notice, $\triangle BQN \sim \triangle NRC$. Thus:

$$\frac{3}{4} = \frac{BN}{NC} = \frac{BM-6}{CM+6} \implies BM = CM = 42.$$

Since $\triangle ABQ \sim \triangle ACR$. Hence:

$$\frac{AQ}{AQ+7} = \frac{AQ}{AR} = \frac{AB}{AC} = \frac{BQ}{CR} = \frac{QN}{NR} = \frac{3}{4} \implies AQ = 21.$$

Now, by Pythagorean Theorem on $\triangle BNQ$, then:

$$\begin{aligned} BQ^2 = BN^2 - NQ^2 &= (BM - MN)^2 - NQ^2 \\ &= (42-6)^2 - 3^2 = 3^2 \cdot 143 \\ &\implies BQ = 3\sqrt{143}. \end{aligned}$$

Now, we may compute the area of $\triangle ABC$:

$$[ABC] = \frac{7}{3} \cdot [ABN] = \frac{7}{3} \cdot \left(\frac{1}{2} \cdot BQ \cdot AN\right) = \frac{7}{6} \cdot 3\sqrt{143} \cdot 24 = \boxed{84\sqrt{143}}.$$

Combinatorics

1. Aaryan, Darsh, Jayden, and Ashvin are playing a game with coins. Each player starts with one coin. If after a turn a player has no coins, then they are out and are no longer playing. Every turn, each player still in the game randomly selects another player who is still playing to give a coin to. Find the probability that after two turns, two players have consolidated all the coins.

Proposed by Aaryan Vaishya

Answer: $\frac{152}{243}$

We consider the multiset of amount of coins each person has. Let A, B, C refer to

$$\{1,1,1,1\} \qquad \{2,1,1\} \qquad \{3,1\}, \{2,2\}$$

respectively.

- Consider $A \to A$. The number of ways this happens is the number of derangements on 4 items, or 9. The probability is $\frac{9}{81}$.
- Consider $A \to B$. The number of ways this happens is $4 \cdot 3 \cdot 2 \cdot 2 = 48$. The probability is $\frac{48}{81}$.
- Consider $A \to C$. The number of ways this happens is $4 \cdot 3 + \binom{4}{2} \cdot 2 = 24$. The probability is $\frac{24}{81}$.
- Consider $B \to C$. The number of ways this happens is $2 + 2 = 4$. The probability is $\frac{4}{8}$.

Then the probability we end at C after 2 moves is

$$\left(\frac{9}{81}\right)\left(\frac{24}{81}\right) + \left(\frac{48}{81}\right)\left(\frac{4}{8}\right) + \left(\frac{24}{81}\right) = \boxed{\frac{152}{243}}$$

2. The positive divisors of 1001 are randomly written on four two-sided cards, one on each face. Find the probability that on every card, one number divides the other.

Proposed by Jacob Khohayting

Answer: $\frac{4}{21}$

Firstly, there are $\binom{8}{4} \cdot 4! \cdot \frac{1}{2^4} = 105$ total ways to label the four cards. Notice, 7, 11, and 13 are all on different cards. Now, we will perform casework:

- If 1 is paired with 1001, then $7 \cdot 11$, $11 \cdot 13$, and $13 \cdot 7$ are all paired with one of 7, 11, and 13. If $7 \cdot 11$ is paired with 7, then $7 \cdot 13$ must be paired with 13 and $11 \cdot 13$ must be paired with 11. A similar result occurs if $7 \cdot 13$ is paired with 7. There are a total of two ways for this case.
- If 1 is not paired with 1001, then since no two of $7 \cdot 11$, $11 \cdot 13$, and $13 \cdot 7$ can be paired with one another, select exactly one of them to be paired with either 1 or 1001. Say, we select $7 \cdot 11$. We will multiply by 3 at the end. There are 2 ways to select the other side to $7 \cdot 11$. Now, if $11 \cdot 13$ pairs with 11, there are 2 ways for $13 \cdot 7$. But, if $11 \cdot 13$ pairs with 13, then $7 \cdot 13$ must pair with 7. Thus, yielding one way. Hence, there are $3 \cdot 2 \cdot (2+1) = 18$ ways for this case.

In total, there are 20 ways. So, the probability that on every card one number divides the other is $\frac{20}{105} = \boxed{\frac{4}{21}}$.

3. Find the number of sequences $s_1, s_2, \ldots, s_{15}$ such that $s_1 = 1$, $s_2 = 2$, $s_3 = 3$, and for all integers $3 < k \leq 15$,

$$s_k \in \{s_{k-1}, s_{k-2}, s_{k-3}\}.$$

Proposed by Jacob Khohayting

Answer: 1581

Define the following values:

- Let A_n be the number of sequences of length n whose final values are of the form $\{a, b, c\}$.
- Let B_n be the number of sequences of length n whose final values are of the form $\{a, a, b\}$.
- Let C_n be the number of sequences of length n whose final values are of the form $\{a, b, a\}$.
- Let D_n be the number of sequences of length n whose final values are of the form $\{b, a, a\}$.
- Let E_n be the number of sequences of length n whose final values are of the form $\{a, a, a\}$.

In search of a recursive formula, consider the following:

- $A_{n+1} = A_n$ since all three terms must be present.
- $B_{n+1} = D_n$ since in order to get (a, a, b), then the last three terms of the n-th sequence is (X, a, a). We require a b to be in this set, so $X = b$ and the last three terms is (b, a, a).
- $C_{n+1} = A_n + B_n + C_n$. We require the previous three terms to be (X, a, b). We require there to be a a in the sequence, so that the next term is a. Clearly, this is satisfied, so $X \in \{a, b, c\}$.
- $D_{n+1} = A_n + B_n + C_n$. We require the previous three terms to be (X, b, a). We require there to be a b in the sequence, so that the next term is b. Clearly, this is satisfied, so $X \in \{a, b, c\}$.
- $E_{n+1} = E_n + D_n$. We require the previous three terms to be (X, a, a). We require there to be a a in the sequence, so that the next term is a. Clearly, this is satisfied, so $X \in \{a, b, c\}$. Note that there is only one D_n since the previous n-th terms are fixed.

Note that $A_3 = 1$ and $B_3 = C_3 = D_3 = E_3 = 0$. In addition,

$$B_{n+1} = D_n = A_{n-1} + B_{n-1} + C_{n-1} = C_n.$$

Therefore, substituting,

$$C_{n+1} = A_n + B_n + C_n = 1 + C_n + C_{n-1}.$$

Notice, we can prove via induction that $C_n = F_{n-1} - 1$ satisfies this recurrence. So, $B_n = F_{n-2} - 1$. Then,

$$E_n = E_3 + \sum_{i=3}^{n-1} D_i = \sum_{i=3}^{n-1} C_i = \sum_{i=3}^{n-1} (F_{i-1} - 1) = \sum_{i=2}^{n-2} (F_i - 1) = (F_n - 2) - (n - 3) = F_n - n + 1.$$

Then, the total sum is equivalent to:

$$\begin{aligned} S_n &= A_n + B_n + C_n + D_n + E_n \\ &= 1 + (F_{n-2} - 1) + (F_{n-1} - 1) + (F_{n-1} - 1) + (F_n - n + 1) \\ &= F_n + 2F_{n-1} + F_{n-2} - n - 1. \\ &= F_{n+2} - n - 1. \end{aligned}$$

Thus, $S_{15} = F_{17} - 15 - 1 = 1597 - 15 - 1 = \boxed{1581}$.

4. Jacob is searching for hooligans on an archipelago of 501 islands, some pairs of which may be connected by a two-way bridge (bridges may overlap). He plans out a route that starts at some island, traverses bridges to visit every island at least once, and returns back to the starting island (note that the starting island is not fixed). He finds that every such route traverses at least M bridges and there are exactly $N > 0$ routes traversing exactly M bridges. Given that $M = 1000$, find the largest possible value of N less than 10^6.

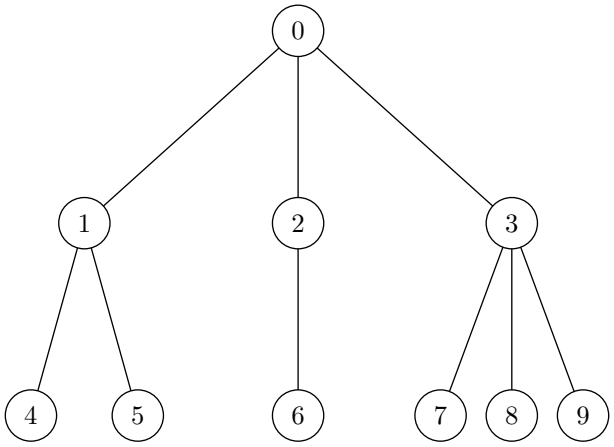
Proposed by Darsh Patel

Answer: 960000

Let G be the graph formed by the archipelago, having $n = 501$ vertices, Firstly, since $N > 0$, G is connected. The first key claim is that the minimum number of edges M in Jacob's route is $2n - 2$ if and only if G is a tree.

Claim: If G is a tree, then $M = 2n - 2$.
Consider the starting vertex v. To visit any other vertex u, we must traverse the unique edge connected to u in the path from v to u. We must again traverse that edge in order to return to v. Thus we traverse every edge at least twice and traverse a total of at least $2n - 2$ edges. Now let G be rooted at v. We will now describe all routes starting at v traversing exactly $2n - 2$ edges (and thus show that some route exists). The first time we get to any vertex u (starting with $u = v$), we implement the following procedure: visit u and all descendants of u, then return to u. We re-implement this same procedure recursively if we travel to a new vertex for the first time while executing it. Note that this is essentially forced, because we cannot return to a descendant of u after moving up from u without visiting the edge starting at u towards v more than twice. This also guarantees that we traverse that edge exactly twice. This applies for every edge, so we traverse exactly $2n - 2$ edges.

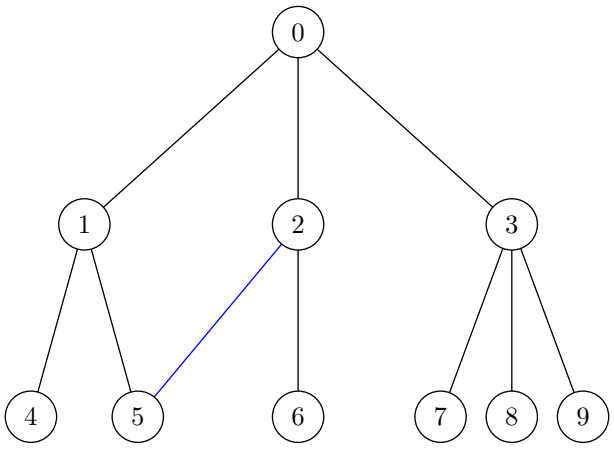
We describe such paths of G slightly better. Once we get to u, its descendants are contained beyond the edges starting at u away from v. We must go down each of these edges, visit all descendants of u in that branch, and return to u, repeating over all edges in any order.



An example of a route is 0141510262037383930

Claim: If G is not a tree, then $M < 2n - 2$.
Take any spanning subtree T of G. Pick a vertex v in some cycle of G. Let a, b be vertices of that cycle that

are not connected in $T \setminus \{v\}$. Now, by the preceding claim we can find a route that visits every vertex of G in 1000 edges by considering T. We will do this in such a way that the branches of T containing a, b are the first two visited, in that order. We also stipulate that a and its descendants are visited last within their branch, and the first edges visited in the branch of b are exactly the path from v to b. The former condition implies that the last edges visited in the branch of a are exactly the path from a to v. We now modify this route by replacing the paths $a \to v \to b$ by the single edge ab from G. Since every vertex not a leaf was visited at least twice in our original route, the vertices between a and v and between v and b will still be visited. This new route thus visits every vertex of G in less than $2n - 2$ edges.



We can replace 5102 with 52

Claim: G **is a tree.**
Since $M = 1000 = 2n - 2$, this follows.

We will now calculate N in terms of G.

Claim: If the vertices of G **are** $v_1, v_2, \ldots, v_n$**, then** $N = (2n-2)(\deg(v_1)-1)!(\deg(v_2)-1)!\ldots(\deg(v_n)-1)!$
First consider routes starting from v_1. Whenever we first any vertex v_k for the first time, as we noted before we can choose to go down each of the edges starting at v_k away from v_1 in any order. This gives $\deg(v_1)!$ possibilities if $v_k = v_1$ and $(\deg(v_k)-1)!$ possibilities if $v_k \neq v_1$. To get the total number of routes, we multiply over all v_k. Summing over all possible starting vertices, we find

$$\begin{aligned} N &= (\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n))(\deg(v_1)-1)!(\deg(v_2)-1)!\ldots(\deg(v_n)-1)! \\ &= (2n-2)(\deg(v_1)-1)!(\deg(v_2)-1)!\ldots(\deg(v_n)-1)! \end{aligned}$$

Let $x_k = (\deg(v_k)-1)!$. We have that $x_1 + x_2 + \cdots + x_n = n - 2$, and further that given any nonnegative integers $x_1, x_2, \ldots, x_n$ with sum $n-2$, we can find a tree G with degrees $x_1+1, x_2+1, \ldots, x_n+1$. Thus we want to find the largest possible value of

$$1000 \cdot x_1!x_2!\ldots x_n!$$

less than 10^6. We note $(2!)^3 5! = 960$, and that no integer in $(960, 1000)$ can be expressed as a product of factorials. Thus the desired value is $\boxed{960000}$.