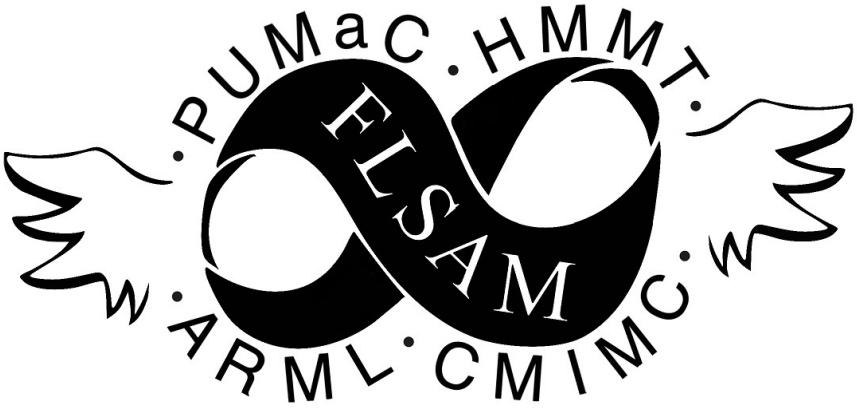
Florida Student Association of Mathematics



2026 PUMaC Tryout

October 10 – 11, 2026

Solutions

Algebra Answers

1. $\left(\frac{51}{4}, \frac{50}{3}\right)$
2. 145π
3. $-\sqrt{30}$
4. $\frac{19}{405}$

Geometry Answers

1. $\frac{396}{5}$
2. $630\sqrt{5}$
3. $13\sqrt{3} + 13\sqrt{35}$
4. $357\sqrt{5}$

Combinatorics Answers

1. 1800
2. 316
3. 1764
4. 6902

Number Theory Answers

1. 108
2. 4340
3. 51
4. 3942

Algebra

1. Compute the unique pair of real numbers (a, b) such that

$$\begin{aligned} \lfloor a \rfloor \{b\} &= 8, \\ \{a\} \lfloor b \rfloor &= 12, \text{ and} \\ \lfloor ab \rfloor \{ab\} &= 106. \end{aligned}$$

(Recall, $\lfloor r \rfloor$ is the largest integer less than or equal to r and $\{r\}$ is the fractional part of r.)

Proposed by Jacob Khohayting

Answer: $\left(\frac{51}{4}, \frac{50}{3}\right)$

Write

$$\begin{aligned} a &= \lfloor a \rfloor + \{a\} = \lfloor a \rfloor + \frac{12}{\lfloor b \rfloor}, \text{ and} \\ b &= \lfloor b \rfloor + \{b\} = \lfloor b \rfloor + \frac{8}{\lfloor a \rfloor}. \end{aligned}$$

Note that $12 < \lfloor b \rfloor$ and $8 < \lfloor a \rfloor$. Therefore,

$$ab = \lfloor a \rfloor \lfloor b \rfloor + 8 + 12 + \frac{96}{\lfloor a \rfloor \lfloor b \rfloor} = \lfloor a \rfloor \lfloor b \rfloor + 20 + \frac{96}{\lfloor a \rfloor \lfloor b \rfloor}.$$

Note that $96 < \lfloor a \rfloor \lfloor b \rfloor$. Therefore,

$$\begin{aligned} \{ab\} &= \frac{96}{\lfloor a \rfloor \lfloor b \rfloor}, \text{ and} \\ \lfloor ab \rfloor &= \lfloor a \rfloor \lfloor b \rfloor + 20. \end{aligned}$$

Multiplying,

$$106 = \lfloor ab \rfloor \{ab\} = (\lfloor a \rfloor \lfloor b \rfloor + 20) \cdot \left(\frac{96}{\lfloor a \rfloor \lfloor b \rfloor}\right) = 96 + \frac{96 \cdot 20}{\lfloor a \rfloor \lfloor b \rfloor}.$$

Thus, $\lfloor a \rfloor \lfloor b \rfloor = 192$. A unique solution with $8 < \lfloor a \rfloor$ and $12 < \lfloor b \rfloor$ occurs at $(\lfloor a \rfloor, \lfloor b \rfloor) = (12, 16)$. So,

$$\begin{aligned} a &= \lfloor a \rfloor + \{a\} = \lfloor a \rfloor + \frac{12}{\lfloor b \rfloor} = 12 + \frac{12}{16} = \frac{51}{4} \\ b &= \lfloor a \rfloor + \{a\} = \lfloor a \rfloor + \frac{12}{\lfloor b \rfloor} = 16 + \frac{8}{12} = \frac{50}{3}. \end{aligned}$$

Therefore, $(a, b) = \boxed{\left(\frac{51}{4}, \frac{50}{3}\right)}$.

2. The graphs of $y = x^2$ and $y = x^3 + 6x^2 - 8$ intersect at three distinct points P, Q, and R. Compute the area of the circumcircle of PQR.

Proposed by Jacob Khohayting

Answer: 145π

Let P, Q, and R have x-coordinates of r, s, and t, respectively. Therefore, r, s, and t are roots of the polynomial:

$$\left(x^3+6x^2-8\right)-\left(x^2\right)=x^3+5x^2-8.$$

Let the circumcircle of PQR be $(x-a)^2+(y-b)^2=c^2$. As P, Q, and R lie on $y=x^2$, then r, s, and t must be roots of the polynomial

$$\begin{aligned}(x-a)^2+\left(x^2-b\right)^2&=c^2\\ x^4+x^2\left(1-2b\right)+x(-2a)+\left(b^2+a^2-c^2\right)&=0.\end{aligned}$$

So, as x^3+5x^2-8 divides the polynomial and the coefficient of x^3 is 0, then 5 must be the other root. Expanding,

$$\left(x^3+5x^2-8\right)(x-5)=x^4-25x^2-8x+40.$$

Comparing coefficients, $b=13$ and $a=4$. Thus,

$$\begin{aligned}b^2+a^2-c^2&=40\\ c^2&=13^2+4^2-40\\ c^2&=145.\end{aligned}$$

Therefore, the area of the circumcircle is $\boxed{145\pi}$.

3. Real numbers a, b, and c are such that

$$\begin{aligned}&(a+b)(a-c)=40,\\ &(b+c)(b-a)=40, \text{ and}\\ &(c+a)(c-b)=40.\end{aligned}$$

Compute the smallest possible value of $a+b+c$.

Proposed by Jacob Khohayting

Answer: $-\sqrt{30}$

Let $x=a+b$, $y=b+c$, and $z=c+a$. We wish to compute for $\frac{x+y+z}{2}$.
Substituting,

$$\begin{aligned}&(z-y)y=-40,\\ &(x-z)z=-40, \text{ and}\\ &(y-x)x=-40.\end{aligned}$$

Evidently, $x,y,z\neq 0$. It follows:

$$0=(z-y)+(x-z)+(y-x)=-40\cdot\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Hence, $xy + yz + zx = 0$. In addition,

$$(z-y)y + (x-z)z + (y-x)x = (xy+yz+zx) - (x^2+y^2+z^2) = -120.$$

Therefore, $x^2 + y^2 + z^2 = 120$. Thus,

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = 120.$$

And so, $x + y + z = \pm\sqrt{120} = \pm 2\sqrt{30}$. So, the smallest possible value is $a + b + c = \boxed{-\sqrt{30}}$.

4. For a positive integer n, let $d(n)$ be the number of ordered triplets (a, b, c) of positive integers such that

$$a + 2b + 3c = n$$

and $\gcd(a, b, c) = 1$. Compute

$$\sum_{n=1}^{\infty} \frac{d(n)}{2^n + 1}.$$

Proposed by Jacob Khohayting

Answer: $\dfrac{19}{405}$

Let $x = \frac{1}{2}$. Notice,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(d(n)\cdot\frac{1}{x^n+1}\right) &= \sum_{n=1}^{\infty}\left(d(n)\cdot\frac{x^{-n}}{1-(-x^{-n})}\right)\\ &= \sum_{n\geq 1} d(n)\cdot\left(\sum_{k\geq 1}(-1)^{k-1}\cdot x^{nk}\right)\\ &= \sum_{m\geq 1} x^m\left(\sum_{k\mid m}(-1)^{k-1}\cdot d\left(\frac{m}{k}\right)\right).\end{aligned}$$

Note that a triple (a, b, c) with $a + b + c = m$ exists only in $d\left(\frac{m}{k}\right)$ when $\gcd(a, b, c) = k$. Hence, set

$$S(x) = \sum_{m\geq 1} x^m\left(\sum_{k\mid m}(-1)^{k-1}\cdot d\left(\frac{m}{k}\right)\right) = \sum_{a,b,c\geq 1} x^{a+2b+3c}(-1)^{\gcd(a,b,c)-1}.$$

We split into two cases:

- If $2 \mid \gcd(a, b, c)$, then this is equivalent to $|S(x^2)|$ after substituting $(a, b, c) = (2a', 2b', 2c')$.
- If no such restrictions exist, then this is equivalent to $|S(x)|$.

Notice, this desired sum is equivalent to $|S(x)| - 2 \cdot |S(x^2)|$. Thus,

$$|S(x)| = \sum_{a,b,c\geq 1} x^{a+2b+3c} = \frac{x}{1-x}\cdot\frac{x^2}{1-x^2}\cdot\frac{x^3}{1-x^3}.$$

Plugging in $x = \frac{1}{2}$ implies

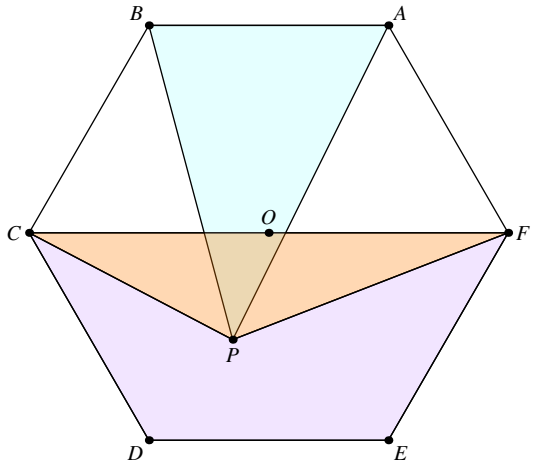
$$\left|S\left(\frac{1}{2}\right)\right| - 2\cdot\left|S\left(\frac{1}{4}\right)\right| = 1\cdot\frac{1}{3}\cdot\frac{1}{7} - 2\cdot\frac{1}{3}\cdot\frac{1}{15}\cdot\frac{1}{63} = \boxed{\frac{19}{405}}.$$

Geometry

1. Point P lies inside regular hexagon $ABCDEF$ such that triangle ABP and pentagon $CDEFP$ have areas 20 and 26, respectively. Compute the area of $ABCDEF$.

Proposed by Jacob Khohayting

Answer: $\frac{396}{5}$



Let $X = [CPF]$, then

$$[COD] + [DOE] + [EOF] = [CDEF] = [CDEFP] + [CPF] = 26 + X.$$

Similarly,

$$[AOB] = [ABP] - \frac{[CPF]}{2} = 20 - \frac{X}{2}.$$

Therefore,

$$20 - \frac{X}{2} = \frac{26+X}{3} = \frac{26}{3} + \frac{X}{3} \implies \frac{34}{3} = \frac{5X}{6} \implies X = \frac{68}{5}.$$

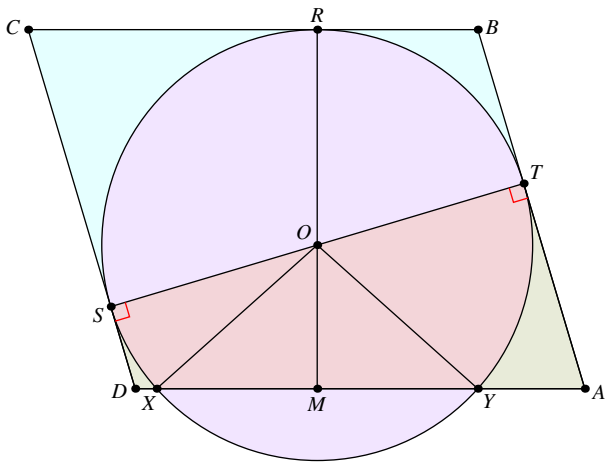
Therefore,

$$[ABCDEF] = [CDEF] \cdot 2 = 2 \cdot (26 + X) = 2 \cdot \frac{198}{5} = \boxed{\frac{396}{5}}.$$

2. Let $ABCD$ be a parallelogram. A circle is tangent to sides AB, BC, and CD intersects side DA at points X and Y such that $DX = 2$, $XY = 30$, and $YA = 10$. Compute the area of $ABCD$.

Proposed by Jacob Khohayting

Answer: $630\sqrt{5}$



By Power of a Point at A and D,

$$DS^2 = DX \cdot DY = 2 \cdot 32 = 64 \implies DS = 8, \text{ and}$$
$$AT^2 = AY \cdot AX = 10 \cdot 40 = 400 \implies AT = 20.$$

Due to Pythagorean Theorem on right trapezoids $ADST$, then

$$ST^2 = AD^2 - (TA - SD)^2 = 42^2 - 12^2,$$

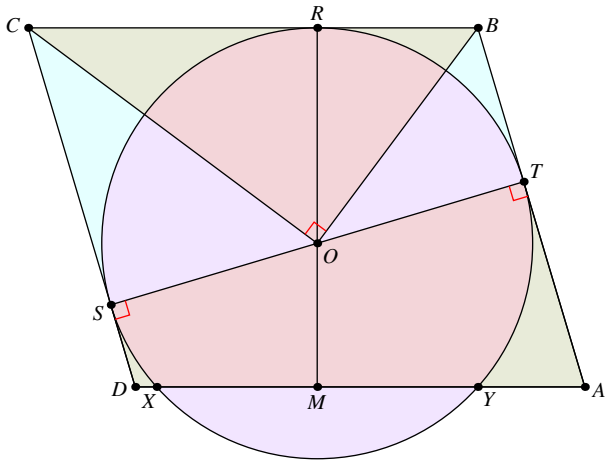
so $ST = 18\sqrt{5}$. As O is the midpoint, then the radius $R = OS = OT = 9\sqrt{5}$. Therefore,

$$OM = \sqrt{R^2 - \left(\frac{XY}{2}\right)^2} = \sqrt{\left(9\sqrt{5}\right)^2 - 15^2} = 6\sqrt{5}.$$

Therefore, $RM = RO + OM = 15\sqrt{5}$. Thus,

$$[ABCD] = RM \cdot AD = 15\sqrt{5} \cdot 42 = \boxed{630\sqrt{5}}.$$

Alternatively, here is a nice way to compute the radius RO (communicated to me by *Joshua Khohayting* and *Arnav Bhatia* separately).



By Power of a Point, $DS = 8$ and $AT = 20$, as mentioned previously. Now, let $CS = x$, then $BT = x - 12$ as $AB = CD$. By Two Tangents Theorem,

$$42 = AD = BC = CS + BT = x + (x - 12) \implies x = 27.$$

So, $CR = 27$ and $BR = 15$. Notice,

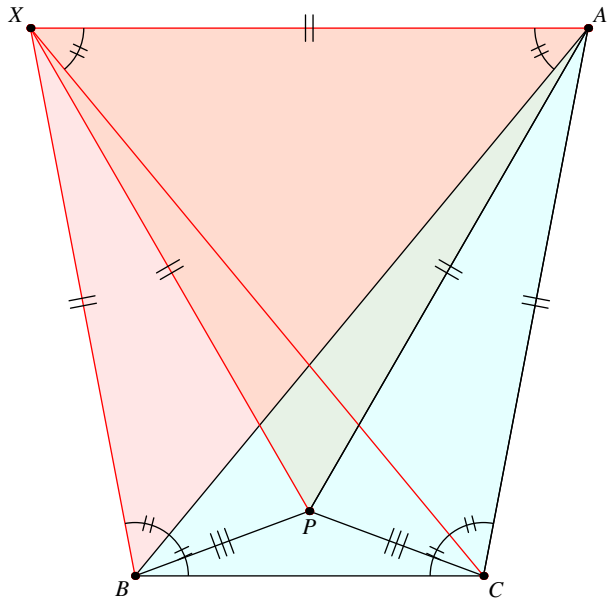
$$\angle COB = 180 - \angle OCB - \angle OBC = 180 - \frac{\angle BCD}{2} - \frac{\angle CBA}{2} = 180 - \frac{180}{2} = 90^\circ.$$

Hence, $RO = \sqrt{27 \cdot 15} = 9\sqrt{5}$. The rest of the solution follows similarly.

3. Let ABC be a triangle such that $\angle ACB = 2\angle ABC$, and let P be a point such that $PB = PC = 6$ and $AP = AC = 18$. Compute the area of triangle ABC.

Proposed by Jacob Khohayting

Answer: $13\sqrt{3} + 13\sqrt{35}$



Let X be the reflection of A over the perpendicular bisector of BC. Therefore,

$$\angle ACX = \angle ACB - \angle XCB = 2\angle ABC - \angle ABC = \angle ABC = \angle XAB = \angle AXC.$$

Therefore, $AX = AC = 18$. Hence, XPA is an equilateral triangle.

Let M be the midpoint of BC and N be the foot of the perpendicular from A to BC. Set $PM = a$ and $NC = b$. By Pythagorean Theorem on MPC and NCA, it follows that

$$a^2 + (9 - b)^2 = 6^2, \text{ and}$$
$$\left(9\sqrt{3} + a\right)^2 + b^2 = 18^2.$$

Expanding, this turns into

$$a^2 + b^2 - 18b + 45 = 0, \text{ and}$$
$$a^2 + b^2 + 18\sqrt{3}a - 81 = 0.$$

Hence, $18\sqrt{3}a - 81 = 45 - 18b$ which turns into $b = 7 - a\sqrt{3}$. Subtitutitng,

$$\left(9\sqrt{3} + a\right)^2 + \left(7 - a\sqrt{3}\right)^2 = 18^2$$
$$4a^2 + 4\sqrt{3}a - 32 = 0$$

By the Quadratic Formula, $a = \dfrac{\pm\sqrt{35}-\sqrt{3}}{2}$, but $a > 0$ implying $a = \dfrac{\sqrt{35}-\sqrt{3}}{2}$. Thus,

$$b = 7 - a\sqrt{3} = \frac{17-\sqrt{105}}{2}.$$

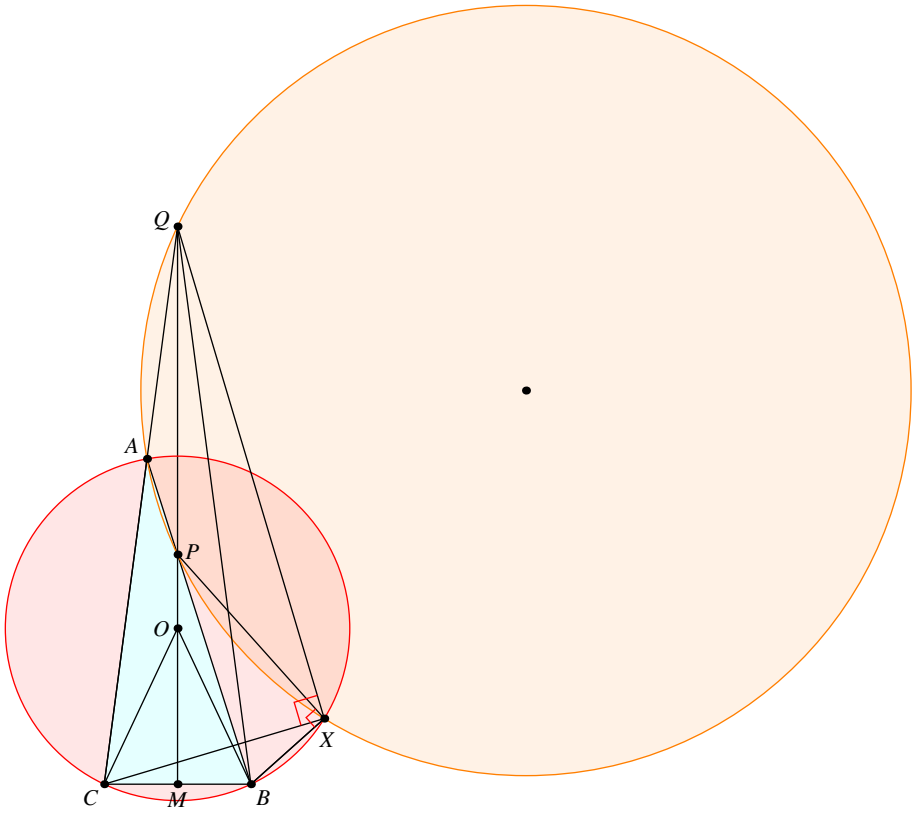
The area of ABC is therefore:

$$\frac{1}{2}\cdot(18-2b)\cdot\left(9\sqrt{3}+a\right) = \frac{1}{2}\cdot\left(1+\sqrt{105}\right)\left(\frac{17\sqrt{3}+\sqrt{35}}{2}\right) = \boxed{13\sqrt{3}+13\sqrt{35}}.$$

4. Let ABC be an acute triangle with circumcircle Ω. The perpendicular bisector of BC meets lines AB and AC at P and Q, and the circumcircle of triangle APQ meets Ω again at $X \neq A$. Given $XB = 18$, $XC = 42$, and $PQ = 60$, find the area of triangle ABC.

Proposed by Jacob Khohayting

Answer: $357\sqrt{5}$



Let M be the midpoint of BC and O be the circumcenter of Ω.

Firstly,

$$\angle QCX = \angle ACX = \angle ABX = \angle PBX$$

and $\angle CQX = \angle XPB$ implying $\triangle CQX \sim \triangle XPB$. In addition,

$$\begin{aligned}\angle QXC &= \angle QXA + \angle AXC \\ &= \angle QPA + \angle ABC \\ &= \angle QPA + (90^\circ - \angle MPB) \\ &= 90^\circ.\end{aligned}$$

Hence, $\angle PXB = 90^\circ$, as well.

Next,

$$\begin{aligned}\angle OQB &= \angle OQC \\ &= 90^\circ - \angle C \\ &= \angle A + \angle B - 90^\circ \\ &= \angle B - (90^\circ - \angle A) \\ &= \angle ABC - \angle OBC \\ &= \angle PBO.\end{aligned}$$

As $\angle POB = \angle BOQ$, then $\triangle POB \sim \triangle BOQ$.

Collectively,

$$\frac{OP}{OB} = \frac{PB}{BQ} = \frac{PB}{QC} = \frac{XB}{XC} = \frac{3}{7}.$$

Therefore, if R is the circumradius, then $OB = R$ and $OP = \frac{3}{7}R$. Then,

$$\frac{PO}{OB} = \frac{BO}{OQ} \implies R^2 = \left(\frac{3}{7}R\right)\left(\frac{3}{7}R + 60\right) \implies 49R = 9R + 1260 \implies R = \frac{63}{2}.$$

Then, $BC = 2R\sin A = 63\sin A$. By Law of Cosines,

$$\begin{aligned}(63 \cdot \sin A)^2 &= 18^2 + 42^2 - 2 \cdot 18 \cdot 42 \cdot \cos A \\ (21 \cdot \sin A)^2 &= 6^2 + 14^2 - 2 \cdot 6 \cdot 14 \cdot \cos A \\ 441 - 441\cos^2 A &= 232 - 168 \cdot \cos A \\ 441\cos^2 A - 168\cos A - 209 &= 0\end{aligned}$$

Solving the quadratic implies $\cos A \in \left\{-\frac{11}{21}, \frac{19}{21}\right\}$. As $\angle A$ is acute, then $\cos A = \frac{19}{21}$. So,

$$\sin A = \sqrt{1^2 - \left(\frac{19}{21}\right)^2} = \frac{4\sqrt{5}}{21}$$

implying $BC = 63\sin A = 12\sqrt{5}$. Also, $MO = \frac{57}{2}$, $OP = \frac{27}{2}$, and $PQ = 60$ implying $MP = 42$ and $MQ = 102$.

Let D be the foot of the altitude from A to BC. Then, as $\triangle ADB \sim \triangle PMB$ and $\triangle ADC \sim \triangle QMC$

$$\frac{AD}{PM} + \frac{AD}{QM} = \frac{BD}{BM} + \frac{CD}{CM} = \frac{BD}{BC/2} + \frac{CD}{BC/2} = 2.$$

Hence,

$$AD = 2 \cdot \frac{PM \cdot QM}{PM + QM} = 2 \cdot \frac{42 \cdot 102}{144} = \frac{119}{2}.$$

Computing,

$$[ABC] = \frac{1}{2} \cdot AD \cdot BC = \frac{1}{2} \cdot \frac{119}{2} \cdot 12\sqrt{5} = \boxed{357\sqrt{5}}.$$

Combinatorics

1. Arnav starts with the string `0112223333`. A *whorl* reverses any three consecutive characters. Compute the number of distinct strings that can be obtained through some sequence of whorls.

Proposed by Jacob Khohayting

Answer: 1800

Equivalently, a whorl reverses any two characters that are two characters apart; that is, for any two digits where there is a digit between them, then they may be swapped. Thus, any digit in an odd position can only be in an odd position at the end while any digit in an even position can only be in an even position at the end. We can break the even positions and the odd positions as follows:

$$(\text{odd}: \ \texttt{01233}) \quad \text{and} \quad (\text{even}: \ \texttt{12233}).$$

Now, we can determine the total number of distinct permutations as:

$$\frac{5!}{2!} \cdot \frac{5!}{2!2!} = \boxed{1800}.$$

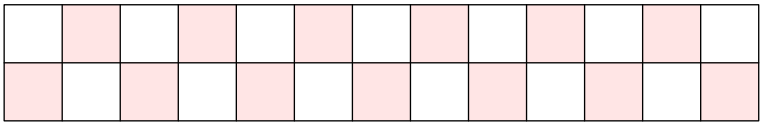
2. Each letter of the alphabet is placed in a distinct cell of a 2×13 rectangular grid. Compute the number of ordered pairs of cells that can contain `a` and `z` such that there exists a sequence of edge-adjacent cells containing `a`, `b`, ..., `z` in order.

Proposed by Jacob Khohayting

Answer: 316

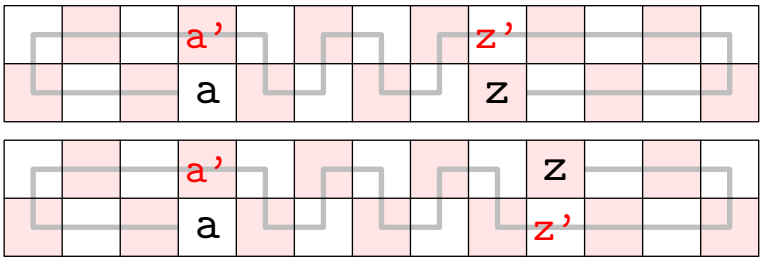
Firstly, notice that for any tiling, evens cannot be placed in adjacent cells and odds cannot be placed in adjacent cells.

- If `a` and `z` are placed in the same column of 2, then this can only occur if `a` and `z` are at the ends of the grid. Otherwise, the grid would only be restricted to one side of `a` and `z` implying the grid would be unfilled. Thus, there are 4 total ways for this scenario.
- Suppose `a` and `z` are not placed in the same column of 2, then place a chessboard tiling on the grid. Then, `a` and `z` must clearly be placed on tiles of distinct colors. We will show pairs of `a` and `z` are valid configurations.



We will outline a process to get from `a` to `z`. Without the loss of generality, suppose `a` is to the left of `z`. Suppose that `a'` is the square in the differing row adjacent to `a` while `z'` is the square in the differing row adjacent to `z`.

First, we will loop around from `a` to `a'`. Notice, there must exist a zig-zag path from `a'` and `z'`. Then, we loop from `z'` to `z` yielding the desired path. An example is shown below describing the path mentioned.



Either case, a path will exist. Therefore, there are $26 \cdot 12 = 312$ total placements in this case.

In total, there are $312 + 4 = \boxed{316}$ total ordered pairs of cells for a and z.

Alternatively, here is a neat solution using recursion (communicated to me by *Arnav Bhatia*). It isn't too difficult to see that if there exists a Hamiltonian path between two points, it must be unique. Suppose the number of Hamiltonian paths on a $2 \times n$ grid is a_n. Notice, $a_1 = 2$.

3. Four alligators, two dolphins and four crocodiles swim in a line, in that order from front to back. Each second, two adjacent animals swap places. After the fewest possible seconds, the four crocodiles are at the front and the four alligators at the back. Each animal records, in order, which animals it swapped with. Compute the number of different collections of ten records that are possible.

Proposed by Jacob Khohayting

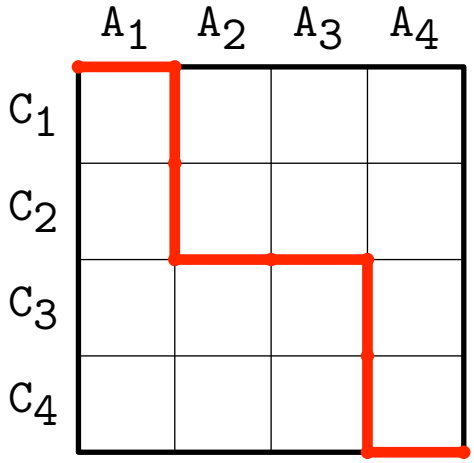
Answer: 1764

Suppose we have some arbitrary record for each dolphin. We claim that given the records for the dolphins, we can uniquely determine the records for the alligators and crocodiles.

Consider an alligator A_i. Note the following

- A_i will never pass any other alligator.
- A_i will always pass D_j and D_k in a particular pre-set order as the dolphins never swap.
- Note that the alligators records will be some permutation of $C_1C_2C_3C_4D_1D_2$ where the dolphins are always in the same order.
- If A_i passes D_k before C_j, then D_k passes A_i before C_j and if A_i passes C_j before D_k, then D_k passes C_j before A_i.

So, the record of the alligator will be predetermined by the record of the dolphin. We only care of the records of dolphins now. Ignore the second dolphin. Below is an example of a dolphin record:



Here, the dolphin passes $\mathtt{A}_1$, then $\mathtt{C}_1$, then $\mathtt{C}_2$, and so on. Therefore, we can visualize the dolphin's records on this grid. If we consider two dolphin's, then this is equivalent to two paths. If the two path's cross, then the dolphin's must have crossed as well, which is unideal as we only take the fewest amount of time. Therefore, we wish to compute the number of non-crossing pairs of paths on a 4×4 grid. Note, the total amount of pairs of paths is

$$\binom{8}{4}^2 = \binom{8}{4}\binom{8}{4} = 4900.$$

Now, we wish to compute all bad pairs of paths. Consider the following bijection: let D_1 (first dolphin) have a path from $(0,0)$ to $(4,4)$ and let D_2 (second dolphin) have a path from $(-1,1)$ to $(3,5)$. Here, we simply shift the path of the second dolphin by the vector $(-1,1)$. Notice, any "good" path initially will now be entirely disjoint; that is, their vertices do not even intersect. Hence, in any pair of bad paths on the shifted perspective, then their vertices must intersect. Reflecting the ends of these paths result in two paths: one path from $(-1,1)$ to $(4,4)$ and another path path from $(0,0)$ to $(3,5)$. Thus, the number of "bad" paths is

$$\binom{8}{3}^2 = \binom{8}{3}\binom{8}{5} = 3136.$$

Therefore, the answer is

$$\binom{8}{4}^2 - \binom{8}{3}^2 = 70^2 - 56^2 = \boxed{1764}.$$

4. Every cell of a 4×4 grid is initially colored white. A *move* consists of choosing a row or column and coloring all four of its cells either all black or all white. Call a coloring of the grid *peelable* if it can be reached by some sequence of moves. Compute the number of peelable colorings.

Proposed by Jacob Khohayting

Answer: 6902

Firstly, note that it is impossible for there to exist rows i_1 and i_2 with columns j_1 and j_2 such that

	j_1	j_2
i_1	W	B
i_2	B	W

	j_1	j_2
i_1	B	W
i_2	W	B

This result is due to the fact that at least one of the four rows/columns above must be colored last.

Consider the following example grid obtainable from $(i_3, \mathtt{B}) \to (j_2, \mathtt{W}) \to (i_2, \mathtt{B}) \to (j_1, \mathtt{B}) \to (j_4, \mathtt{B})$:

	j_1	j_2	j_3	j_4
i_1	B	W	W	B
i_2	B	B	B	B
i_3	B	W	B	B
i_4	B	W	W	B

For each row, the set of elements that are white are: $i_1 : \{j_2, j_3\}$, $i_2 : \{\}$, $i_3 : \{j_2\}$, and $i_4 : \{j_2, j_3\}$. The key claim is that for any two rows $i_{\sigma(1)}$ and $i_{\sigma(2)}$, one must contain the other. Suppose not, then if there exists some element $e_1 \in i_{\sigma(1)}$ and $e_1 \notin i_{\sigma(2)}$ and another element $e_2 \in i_{\sigma(2)}$ and $e_2 \notin i_{\{\sigma(1)\}}$. But, this would essentially be a checkerboard among these two rows and columns, a contradiction, as we showed earlier. Hence, for every pair of sets, one must be a subset of the other.

Now, this means we can rearrange the columns and rows to get the following form:

	j_2	j_3	j_1	j_4
i_1	W	W	B	B
i_4	W	W	B	B
i_3	W	B	B	B
i_2	B	B	B	B

It is clear of the process we take from here:

- Start with a black grid (starting with white or black doesn't matter).
- Color the row(s) i with the minimum number of cells entirely W. Color the columns black so those rows are perfectly completely colored as desired.
- Repeat on the next row(s) i with the next minimum number of cells entirely W, and so on.

This is equivalent to partitioning the rows and columns into an equal number of sets. In this case, we partitioned the rows and columns into the following sets: $\{\{i_3\}, \{i_1, i_4\}, \{i_2\}\}$ which is the rows colored at each step (the last one is all-black columns and this can be empty) and $\{\{j_2\}, \{j_3\}, \{j_1, j_4\}\}$ are the rows which stop getting blackened at each step (we can have all-white rows and this set of white rows can be empty as seen in this example). Every step must contain at least one row and at least one column otherwise it isn't a step.

As a result, this is equivalent to

$$\sum_{k=0}^{4} N(4,k)^2$$

where $N(4,k)$ is the number of ways to place 4 objects into k ordered non-empty groups or possibly also in one other group that can be empty.

Note that

$$N(a,b) = (b+1) \cdot N(a-1,b) + b \cdot N(a-1,b-1).$$

Computing out the recurrence table with $N(a,0) = 1$ for all a and $N(0,b) = 0$ for $b \geq 1$, then

$$N(4,0)^2 + N(4,1)^2 + N(4,2)^2 + N(4,3)^2 + N(4,4)^2 = 1^2 + 15^2 + 50^2 + 60^2 + 24^2 = \boxed{6902}.$$

Number Theory

1. A positive integer n is such that $\frac{n}{24}$ has 36 positive integer divisors and $\frac{n}{36}$ has 24 positive integers divisors. Compute the number of positive integers divisors of n.

Proposed by Jacob Khohayting

Answer: 108

Let $N = 2^a 3^b M$ where $\gcd(M, 6) = 1$. Then,

$$36 = \tau\left(\tfrac{N}{24}\right) = (a-2)b \cdot \tau(M), \text{ and}$$
$$24 = \tau\left(\tfrac{N}{36}\right) = (a-1)(b-1) \cdot \tau(M).$$

Therefore,

$$\begin{aligned} 24 \cdot \big((a-2)b \cdot \tau(M)\big) &= 36 \cdot \big((a-1)(b-1) \cdot \tau(M)\big) \\ 2ab - 4b &= 3ab - 3a - 3b + 3 \\ 0 &= ab - 3a + b + 3 \\ -6 &= (a+1)(b-3) \end{aligned}$$

Note that $a+1 \geq 4$ so $a+1 = 6$ or $a = 5$. Thus, $b-3 = -1$ and $b = 2$. Hence,

$$\tau(M) = \frac{36}{(a-2)b} = \frac{36}{3 \cdot 2} = 6.$$

Therefore, $\tau(N) = (a+1)(b+1) \cdot \tau(M) = 6 \cdot 3 \cdot 6 = \boxed{108}$.

2. Compute the *third* smallest positive integer N for which

- N has two divisors with units digit 7,
- N has four divisors with units digit 4, and
- N has eight divisors with units digit 0.

Proposed by Jacob Khohayting

Answer: 4340

Let $N = 2^a 5^b M$ where $\gcd(M, 10) = 1$. For $r \in \{1, 2, 3, 4\}$, define D_r to be the number of divisors which are equivalent to $r \pmod 5$. Note that

$$t = \tau(M) = D_1 + D_2 + D_3 + D_4.$$

Note the following observations:

- As eight divisors have units digit 0, then $abt = 8$.
- As two divisors have units digit 7, then the divisor must not be divisible by 2 or 5, but must be $2 \pmod 5$. Hence, $D_2 = 2$.
- A number with units digit 4 must take on the form $2^k s$ where $\gcd(s, 10) = 1$. Summing as k grows from 1 to a this, therefore, is the first a terms of the list D_2, D_1, D_3, D_4, D_2, and so on.

As $D_2 = 2$ and $D_1 \geq 1$, then $t \geq 3$. Hence, as $t \mid 8$, then $t = 4$ or $t = 8$. We have two cases:

- If $t = 8$, then $a = b = 1$. So, as $a = 1$, the number of integers with units digit 4 is $D_2 = 2$, contradiction.
- Thus, $t = 4$. We just saw $a = 1$ is impossible. So, $a = 2$ and $b = 1$. Thus, $D_2 + D_1 = 4$ implying $D_1 = 2$.

Hence, $D_1 = 2$ and $D_2 = 2$. Indeed, $t = 4$ and so M has exactly four divisors. We have two cases:

- If m takes on the form of p^3, then

  $$\{p^0, p^1, p^2, p^3\} \equiv \{1, 1, 2, 2\} \pmod 5.$$

  It isn't too hard to see this is impossible.
- If m takes on the form of pq, then

  $$\{1, p, q, pq\} \equiv \{1, 1, 2, 2\} \pmod 5.$$

  This occurs if $p \equiv 1 \pmod 5$ and $q \equiv 2 \pmod 5$.

As p and q are odd, then $p \equiv 1 \pmod{10}$ and $q = 7 \pmod{10}$. Now, it suffices to minimize the product of pq. The possible values for p and q are:

$$p \in \{11, 31, 41, 61, \ldots\},$$
$$q \in \{7, 17, 37, 47, \ldots\}.$$

The smallest values of pq are:

$$7 \cdot 11 = 77$$
$$17 \cdot 11 = 187$$
$$7 \cdot 31 = 217$$
$$7 \cdot 41 = 287$$
$$\vdots$$

Therefore, the answer is $2^2 \cdot 5^1 \cdot 7 \cdot 31 = \boxed{4340}$.

3. Compute the number of ordered triplets (a, b, c) of positive integers with $a, b, c \leq 40$ such that

$$\gcd(a, b, c) \operatorname{lcm}(a, b, c) = \frac{ab + bc + ca}{10}.$$

Proposed by Roger He

Answer: 51

Let $d = \gcd(a, b, c)$, and write $a = dx$, $b = dy$, and $c = dz$ where $\gcd(x, y, z) = 1$. The given equation becomes

$$10 \operatorname{lcm}(x, y, z) = xy + yz + zx.$$

Since $\operatorname{lcm}(x, y, z)$ is divisible by each of x, y, z, the equation implies $x \mid yz$, $y \mid zx$, and $z \mid xy$. Let

$$p = \gcd(x, y), \qquad q = \gcd(y, z), \qquad r = \gcd(z, x).$$

Because $\gcd(x, y, z) = 1$, the divisibility conditions imply $x = pr$, $y = pq$, and $z = qr$ where p, q, r are pairwise coprime positive integers. Consequently,

$$\operatorname{lcm}(x, y, z) = pqr.$$

Substituting into the equation gives

$$10pqr = p^2qr + pq^2r + pqr^2.$$

Dividing by pqr, we obtain $p + q + r = 10$. Assume $p \leq q \leq r$. The positive integer solutions to $p + q + r = 10$, with p, q, r pairwise coprime and up to permutation, are

$$(p, q, r) \in \{(1, 1, 8),\ (1, 2, 7),\ (1, 4, 5),\ (2, 3, 5)\}.$$

Using $x = pr,\ y = pq,\ z = qr$, these yield the following normalized triples, up to permutation:

$$(1, 8, 8), \quad (2, 7, 14), \quad (4, 5, 20), \quad (6, 10, 15).$$

Finally, since $a = dx,\ b = dy,\ c = dz$ and $a, b, c \leq 40$, the scaling factor satisfies

$$1 \leq d \leq \left\lfloor \frac{40}{\max(x, y, z)} \right\rfloor.$$

We count the ordered triples in each case:

(x, y, z)	Permutations	Choices of d	Count
$(1, 8, 8)$	3	5	15
$(2, 7, 14)$	6	2	12
$(4, 5, 20)$	6	2	12
$(6, 10, 15)$	6	2	12

Therefore, the total number of ordered triples is $15 + 12 + 12 + 12 = \boxed{51}$.

4. Jerry draws some of the lines of symmetry of a regular 12-gon, and Jacob draws all 12 of them. In either drawing, a token placed anywhere in the plane may be moved, any number of times, to its reflection across any drawn line. Compute the number of choices of Jerry's lines such that, for every point P, a token starting at P can reach exactly the same points in Jerry's drawing as in Jacob's.

Proposed by Roger He

Answer: 3942

Let $n = 12$. Label the 12 lines of symmetry of the regular 12-gon by their angles

$$\ell_k : \theta = \frac{k\pi}{n} \qquad (k = 0, 1, \ldots, n-1),$$

and write $S = \{\ell_0, \ell_1, \ldots, \ell_{n-1}\}$.

If a subset $T \subseteq S$ is chosen, then reflecting repeatedly in lines from T moves a point along the circle of its radius by angles that are integer multiples of

$$\Delta_T := 2 \cdot \gcd\left\{\, |i-j| : \ell_i, \ell_j \in T \,\right\} \cdot \frac{\pi}{n}.$$

(Indeed, reflecting in ℓ_i then in ℓ_j rotates by twice the angle between ℓ_i and ℓ_j, so the smallest positive rotation obtainable is $2 \cdot d \cdot \frac{\pi}{n}$, where d is the greatest common divisor of the index differences for lines in T.)

Thus, the reflections from T produce the same set of points on each circle as the full set S if and only if the smallest positive rotation obtainable from T equals the smallest positive rotation obtainable from S. For S, the smallest rotation is $2 \cdot \frac{\pi}{n}$, so the necessary and sufficient condition is

$$\gcd\{n, |i-j| : \ell_i, \ell_j \in T\,\} = 1.$$

So the problem reduces to: how many nonempty subsets $A \subseteq \{0, 1, \ldots, n-1\}$ have

$$\gcd\{n, |i-j| : i, j \in A\,\} = 1?$$

For a divisor d of n, write $f(d)$ for the number of nonempty subsets all of whose elements are congruent modulo d. Every such subset is contained in one of the d residue classes modulo d, each class having n/d elements, hence

$$f(d) = d\left(2^{n/d} - 1\right).$$

Let $F(g)$ be the number of nonempty subsets whose index-differences have greatest common divisor exactly g. Then for every $d \mid n$, we have the relation

$$f(d) = \sum_{g : d \mid g \mid n} F(g).$$

Möbius inversion gives the formula for $F(1)$:

$$F(1) = \sum_{d|n} \mu(d) f(d) = \sum_{d|n} \mu(d) \cdot d \left(2^{n/d} - 1\right),$$

where μ is the Möbius function.
For $n = 12$, the divisors d with $\mu(d) \neq 0$ are $d = 1, 2, 3, 6$. Evaluating each term yields:

$$\begin{aligned}
d = 1: &\quad \mu(1) = 1, &\quad f(1) &= 1 \cdot (2^{12} - 1) = 4095 \\
d = 2: &\quad \mu(2) = -1, &\quad f(2) &= 2 \cdot (2^{6} - 1) = 126 \\
d = 3: &\quad \mu(3) = -1, &\quad f(3) &= 3 \cdot (2^{4} - 1) = 45 \\
d = 6: &\quad \mu(6) = 1, &\quad f(6) &= 6 \cdot (2^{2} - 1) = 18
\end{aligned}$$

Substituting these into the Möbius inversion sum gives:

$$\begin{aligned}
F(1) &= 1 \cdot (2^{12} - 1) - 2 \cdot (2^{6} - 1) - 3 \cdot (2^{4} - 1) + 6 \cdot (2^{2} - 1) \\
&= 4095 - 126 - 45 + 18 \\
&= 3942.
\end{aligned}$$

Therefore, the number of choices of Jerry's lines is $\boxed{3942}$.