Answer Key

1. 329
2. 17
3. 66
4. 143
5. 69
6. 488
7. 19
8. 911
9. 7
10. 252

1. Andrew chooses two positive divisors of 14^{14} and notes that they sum to 357. What number does he get when he computes the absolute difference of these two numbers?

   Answer: 329

   Proposed by Andrew Xing

   Since 357 is odd, one of our integers must be an odd power of 7. From here, we can test out $1, 7, 49$, and 343 to get that the two numbers are 343 and 14, so the answer is $343 - 14 = \boxed{329}$.

2. In a unit square $ABCD$, points P and Q are chosen on the perimeter such that $PQ = 1$. As P and Q vary, the midpoint of PQ traces out a curve. The area enclosed by this curve can be expressed as $a - \frac{\pi}{b}$, for positive integers a and b. Compute $a^2 + b^2$.

   Answer: 17

   Proposed by Karthik Vedula

   WLOG, let P be on AB and Q be on AC. The distance from the midpoint of PQ to A would be half the hypotenuse of APQ, which is $0.5(1) = 0.5$. Applying this logic to all 4 cases tells us that the distance from this midpoint to the closest vertex of the square is 0.5. This means that the desired locus is the four quarter-circles centered at the vertices of the squares with radius 0.5. The desired area is the entire square minus the area of a circle with radius 0.5, which is $1 - \frac{\pi}{4}$, so the answer is $1^2 + 4^2 = \boxed{17}$.

3. Given that
   $$\lceil x \rceil^2 - 12\lfloor x \rfloor + 20 = 0,$$
   find the sum of all distinct possible values of $\lfloor 2x \rfloor$.

   Answer: 66

   Proposed by Aaron Hu

   If x is an integer, then clearly $x = 2, 10$ are solutions. Otherwise, $\lceil x \rceil = \lfloor x \rfloor + 1$, so the equation rewrites as
   $$0 = \lfloor x \rfloor^2 - 10\lfloor x \rfloor + 21 = (\lfloor x \rfloor - 3)(\lfloor x \rfloor - 7),$$
   so $x \in (3, 4) \cup (7, 8)$ are solutions. Then the answer is just
   $$4 + 20 + 6 + 7 + 14 + 15 = \boxed{66}.$$

4. A frog jumps on lily pads starting at lily pad 1 such that from lily pad n the frog can jump to either lily pad $n+1$ or lily pad $n+2$. Denote A_n as the number of paths the frog can take to reach lily pad n. Compute $A_1 + A_2 + \cdots + A_{10}$.

   Answer: 143

   Proposed by Ahan Mishra

   Since the frog can jump to either of the next two lily pads this provides the recursive formula $A_n = A_{n-1} + A_{n-2}$ for $n \geq 3$. $A_2 = 1$ because the only way to reach the second lily pad is to take a 1 step jump from the first lily pad. Note that A_n is the sequence of Fibonacci numbers, and since $\Sigma_{i=1}^{n} F_n = F_{n+2} - 1$, we have the answer as $A_{10+2} - 1 = 144 - 1 = \boxed{143}$. Alternatively, the sequence A (which goes up to 55 for A_{10} is small enough that it can be added manually.)

5. Jessica fills the 64 squares of an 8×8 table with distinct positive integers from 1 to 66 in a way such that the sum of each row is divisible by 17 and the sum of each column is divisible by 21. What is the sum of the two positive integers she omits?

Answer: 69

Proposed by Jessica Wan

Note that $1+2+\cdots+66 = \frac{66\cdot 67}{2} = 2211$, and that $2211 \equiv 1 \pmod{17}$, $2211 \equiv 6 \pmod{21}$. Therefore $2211 \equiv 69 \pmod{17 \cdot 21}$. On the other hand, the conditions of the problem imply that $2211 - x$ is divisible by $17 \cdot 21$, where x is the sum of the two missing positive integers. As $1+2 \leq x \leq 65+66$, it follows that $x = \boxed{69}$.

6. Aaron has a standard 52 deck of cards. He randomly draws 2 of them, and they're both aces! Now, he randomly distributes the remaining cards by giving 5 cards each to 10 friends. The probability that no one gets more than one ace can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m+n$.

Answer: 488

Proposed by Alex Li

The probability the aces go to the same person is $\frac{\binom{48}{3}}{\binom{50}{5}} = \frac{2}{245}$, so the probability is $\frac{243}{245}$. The answer is $\boxed{488}$.

7. Alex is tired of the boring geometric series, and so he decides to add some spice by developing a probabilistic geometric series! The first term of this series in 1, and every successive term is the previous term divided by 2 or the previous term divided by 3, each occurring with probability $\frac{1}{2}$. The expected value of the sum of this series can be expressed as $\frac{m}{n}$, for relatively prime positive integers m, n. Compute $m+n$.

Answer: 19

Proposed by Karthik Vedula

Let S denote the expected value of the series. If the next term is $\frac{1}{2}$, then the expected value of the sum of the series excluding the first term is $\frac{S}{2}$. Similarly, if the next term is $\frac{1}{3}$, then the expected value excluding the first term would be $\frac{S}{3}$. This means

$$S = 1 + \frac{1}{2}\left(\frac{S}{2} + \frac{S}{3}\right) \implies S = \frac{12}{7}.$$

This means our answer is $12 + 7 = \boxed{19}$.

8. Jack has 6 presents for 6 children, with each present being for exactly 1 child and no 2 presents being for the same child. However he forgot to write down who each present is for after wrapping them! If he randomly gives each child a gift, the probability that more than 1 child gets their intended gift can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. What is $m+n$?

Answer: 911

Proposed by Xuzhou Ren

We use complementary counting. No child gets their intended gift: $!6 = 265$ ways. Exactly 1 child gets their intended gift: $\binom{6}{1} \cdot !5 = 6 \cdot 44$ ways. So the probability is $1 - \frac{529}{720} = \frac{191}{720}$, and the answer is $191 + 720 = \boxed{911}$.

9. Let $a_1, a_2, \ldots, a_n$ be an arithmetic sequence with first term 1, and let $g_1, g_2, \ldots, g_n$ be a geometric sequence with first term 1, where all terms of both sequences are integers. Let $s_1, s_2, \ldots, s_n$ be the sequence with $s_k = a_k + g_k$. What is the smallest integer $k > 1$ such that s_k cannot equal 2022?

Answer: 7

Proposed by Tom Zhang

Let m denote the common difference of the arithmetic sequence and p denote common ratio of the geometric sequence. Suppose that $k = n + 1$ is the smallest integer not satisfying the condition. We have

$$a_k + g_k = (nm + 1) + p^n = 2022 \implies nm + p^n = 2021.$$

We don't need to care about $n = 1, 2, 5$ because they will definitely satisfy the condition. If $n = 3$, then

$$3m + p^3 = 2021 \implies p^3 \equiv 2 \pmod 3,$$

so $p = 2$ satisfies. If $n = 4$, then $4m + p^4 = 2021$, so $p = 1$ satisfies. If $n = 6$, then

$$6m + p^6 = 2021 \implies p^6 \equiv -1 \pmod 6,$$

which is not possible. So the smallest such k is $6 + 1 = \boxed{7}$.

10. Let $ABCD$ be a trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = 13$, $BC = 14$, $CD = 15$. Given that the B-median in $\triangle ABC$ and the C-median in $\triangle BCD$ intersect on $\overline{AD}$, find the area of $ABCD$.

Answer: 252

Proposed by Aaron Hu

Let M, N denote the midpoints of $\overline{AC}, \overline{BD}$, respectively, and let P denote the concurrency point mentioned in the problem statement. Then $\triangle MAP \cong \triangle MCB$ and $\triangle NDP \cong \triangle NBC$, so

$$AD = AP + PD = 2BC = 28.$$

From here, we readily find that the trapezoid has height 12, so the answer is $\frac{1}{2}(12)(14 + 28) = \boxed{252}$.