Florida Student Association of Mathematics

2022 Spooky Halloween Contest Boss Fight Solutions

October-November 2022

1. **[4]** KitKats can be approximated as trapezoidal prisms while York candies can be approximated as cylinders. KitKats consist of 4 prisms, with the trapezoidal cross section having bases of lengths 12mm and 16mm, and a height of 10mm. The lengths of the prisms are 90mm. Yorks have heights of 10mm and radii of 24mm. The ratio of the volume of a KitKat to a York can be expressed as simple fraction $\frac{m}{n\pi}$. Compute $m+n$.

   Answer: 39

   Proposed by Yuhan Niu

   The formula for the volume of the KitKat is $4l\frac{h(b_1+b_2)}{2}$. The formula for the York is $h\pi r^2$. Plugging in numbers and dividing results in $\frac{35}{4\pi}$. Therefore, $35+4=\boxed{39}$.

2. **[4]** A scary number frightens Andrew if, in base 10, it is less than 1024 and greater than 0, but in base 16, the sum of its digits is divisible by 5. Naturally, Aaron writes down every possible scary number. How many numbers does Aaron write?

   Answer: 204

   Proposed by Andrew Xing

   We start by looking at our second condition. Because 16 is 1 mod 5, our second condition essentially states that our number must be divisible by 5. There are $\frac{1020}{5}=204$ numbers from 1 to 1024 that are divisible by 5, so our answer is $\boxed{204}$.

3. **[4]** Skeletons numbered 1 through 10 are racing around a circular track starting at the 0 meter mark. Skeleton n goes at a speed of n meters per second and the circumference of the track is 99 meters. A skeleton trap activates at the 50 meter mark every 20 seconds. If the skeleton's run around the track forever, how many skeletons are eventually trapped?

   Answer: 7

   Proposed by Ahan Mishra

   In order for the skeleton to get to the trap location, the skeleton's time of travel t must satisfy the equation $50+99k=n\cdot t$, where k is some integer that represents the number of laps. For the trap to activate at the proper time, $t=20v$ for some integer v. So the equation becomes $50+99k=20vn$. Now note that skeleton 3, skeleton 6 and skeleton 9 can never be trapped, because plugging in $n=3$, $n=6$ and $n=9$ would result in the right side being a multiple of 3 while the left side is not a multiple of 3. For all other skeleton numbers, $20n$ and 99 are relatively prime. This means that all the other skeletons will be trapped. Specifically, taking both sides $\pmod{99}$, we obtain $20nv\equiv 50 \pmod{99}$.

   Modular inverses exist for relatively prime numbers, so the modular inverse of $20n$ can be multiplied on both sides of the equivalence to obtain solutions for v immediately, which can then be plugged back in for k. So the final number of trapped skeletons is $10-3=\boxed{7}$.

4. **[4]** Compute the sum of all possible positive integers $\overline{SKUL}$ satisfying $\overline{SKUL}=\overline{UL}^2$, where S,K,U,L are (not necessarily distinct) digits not all equal to 0.

   Answer: 6402

   Proposed by Tom Zhang

   Call $\overline{UL}=x$; then the problem is simply asking for solutions to

$$x(x-1)=x^2\equiv x \pmod{100},$$

where $0 < x < 100$. Clearly, $x = 1$ is a solution, and for other solutions we need to make sure that one of $x, x-1$ is the multiple of 25 and the other is the multiple of 4. By simple case check, $x = 25, 76$ work, so the answer is $1^2 + 25^2 + 76^2 = \boxed{6402}$.

5. **[6]** There is a 60% chance that Gerber sees a skull on Halloween. If he sees a skull, he loses focus and only collects half as many candies as if he had not seen a skull. The expected number of candies Gerber collects is 56. If he takes a different route so that there is only a 30% chance of seeing a skull, how many pieces of candy can Gerber now expect to collect? Assume that the number of candies collected given that Gerber sees a skull and the number given he doesn't see a skull remain the same as before.

Answer: 68

Proposed by Ahan Mishra

The equation for the original expected number of candies is $3/5s + 2/5n = 56$, where s represents the number of candies given that Gerber sees a skull and n represents the number of candies given that he does not see a skull. We also have that $n = 2s$, so $3/5s + 4/5s = 7/5s = 56$. Then, $s = 56 \cdot 5/7 = 8 \cdot 5 = 40$, and $n = 2s = 80$. Under the new route, the expected number of candies will be $40 \cdot 0.3 + 80 \cdot 0.7 = 12 + 56 = \boxed{68}$.

6. **[6]** Jessica is scared of all numbers expressible in the form

$$\frac{abc}{a+b+c},$$

where a, b, c are positive integers. What is the largest integer less than 2022 that she is not scared of?

Answer: 0

Proposed by Jessica Wan

Observe that with $a = 1, b = n+1, c = n(n+2)$, we have:

$$\frac{abc}{a+b+c} = \frac{n(n+1)(n+2)}{(n+1)(n+2)} = n$$

for all $n \in \mathbb{N}$. It's also immediate that when a, b, c are positive integers,

$$\frac{abc}{a+b+c} > 0,$$

which means the largest inexpressible integer is $\boxed{0}$.

7. **[6]** Over all pairs of integers (x, y) with $0 \leq x, y \leq 2022$ satisfying

$$y^{\log_3(y^4)} = x^{\log_{\sqrt{3}}(y^{10}) - \log_9(x^{50})},$$

what is the sum of all possible values of $x + y$?

Answer: 1330

Proposed by Jessica Wan

First, the last equation is equivalent after simplification to:

$$y^{4\log_3(y)} = x^{20\log_3(y) - 25\log_3(x)}.$$

After taking $\log_3$ of both sides we obtain:

$$4\log_3(y) \cdot \log_3(y) = \log_3(x)(20\log_3(y) - 25\log_3(x)),$$

which is equivalent to

$$(2\log_3(y) - 5\log_3(x))^2 = 0 \iff 2\log_3(y) = 5\log_3(x).$$

As $x, y \in \mathbb{N}$, we require $x = n^2, y = n^5$ for some $n \in \mathbb{N}$, which yields the solutions $(1^2, 1^5)$, $(2^2, 2^5)$, $(3^2, 3^5)$, $(4^2, 4^5)$ (noting that $5^5 > 2022$), for an answer of $\boxed{1330}$.

8. [6] Find the sum of the three distinct prime factors of $6^8 - 6^5 + 1$.

Answer: 2471

Proposed by Jessica Wan

We can rewrite this number as

$$\begin{aligned} 6^8 - 6 * 6^4 + 1 &= (6^4)^2 - 6 * 6^4 + 1 \\ &= (6^4 - 1)^2 - 4 * 6^4 \\ &= (6^4 - 1)^2 - (2 * 6^2)^2 \\ &= (6^4 - 2 * 6^2 - 1)(6^4 + 2 * 6^2 - 1) \\ &= 2123 * 2267 = 11 * 193 * 2267 \end{aligned}$$

After finding 11 as the minimal prime divisor of either factor, we are left with three relatively prime values, and thus they are the prime divisors. Their sum is $\boxed{2471}$.

9. [8] There exists a function $f(x) = x^4 - 4x^3 + 3x^2 - 2x + 1$. Let the roots of $f(x)$ be r_1, r_2, r_3, r_4. There exists another function $g(x)$ with 6 roots of the form $r_i + r_j$ where $0 < i < j \leq 4$. Given that $g(x)$ is a monic polynomial of degree 6, find the coefficient of the x^4 term in $g(x)$.

Answer: 54

Proposed by Sailalitha Kodukula

This is just the second symmetric sum of 6 numbers of the form $r_i + r_j$ for $0 < i < j \leq 4$. From Vieta, this is

$$3\sum r_i^2 + 8\sum r_i r_j = 3(4^2 - 2 \cdot 3) + 8 \cdot 3 = 30 + 24 = \boxed{54}.$$

10. [8] In $\triangle ABC$, let M, N denote the midpoints of $\overline{AB}$, $\overline{BC}$, respectively. Suppose that the circumcircle of $\triangle AMN$ is tangent to $\overline{BC}$ and intersects $\overline{AC}$ a second time at $D \neq A$. Given that $AN = 9$ and $BC = 12$, find AD^2.

Answer: 98

Proposed by Aaron Hu

From Power of a Point,

$$AB^2/2 = BM \cdot BA = BN^2 = (12/2)^2 = 36,$$

so $AB = 6\sqrt{2}$. Note that

$$\angle BAN = \angle MAN = \angle MNB = \angle ACB,$$

implying that $\triangle NBA \sim \triangle ABC$. Then $\frac{AC}{NA} = \frac{AB}{NB}$, so

$$AC = \frac{AB \cdot AN}{BN} = \frac{6\sqrt{2} \cdot 9}{6} = 9\sqrt{2}.$$

From Power of a Point again,

$$CD \cdot CA = CN^2 \implies CD = 2\sqrt{2},$$

so $AD = AC - CD = 7\sqrt{2}$, and the answer is $\boxed{98}$.

11. **[8]** Raphael is in a haunted house and he can't get out! The resident Zombie gives Raphael the following problem: $2^{22}+1$ has 3 distinct prime factors, denoted as p_1, p_2, and p_3, where $p_1 \leq p_2 \leq p_3$. In order to get our of the haunted house, Raphael has to take p_2 steps. How many steps should he take?

Answer: 397

Proposed by Ahan Mishra

$2^{22} = 2^2 \cdot 2^{20} = 4 \cdot (2^5)^4$, so we can apply the Sophie Germain identity, $4a^4 + b^4 = (2a^2+b^2)^2 - 4a^2b^2 = (2a^2-2ab+b^2)(2a^2+2ab+b^2)$. Specifically, $a = 2^5 = 32$ and $b = 1$. The identity produces $2^{22}+1 = (2\cdot 1024 - 2\cdot 32 + 1)(2 \cdot 1024 + 2 \cdot 32 + 1) = 1985 \cdot 2113$. $1985/5 = 397$ so $2^{22}+1 = 5 \cdot 397 \cdot 2113$. A cursory divisibility check with primes under 20 shows that 397 is prime, and 2113 is not a multiple of 397. It is given that $2^{22}+1$ has three distinct prime factors so this means that $p_3 = 2113$, $p_1 = 5$, and $p_2 = \boxed{397}$.

12. **[8]** 5 people in costumes play a game. In a round, each person randomly chooses to unmask one other person, and those who have been revealed lose. The game ends when no more legal moves can be made, i.e. no one can unmask another player still in the game. Given the probability that there is exactly one winner equals $\frac{m}{n}$ for relatively prime positive integers m, n, find $m+n$.

Answer: 47

Proposed by Jessica Wan

Notice each round will reveal at least 2 people. So, for a 5 person game, there are two ways to end up with 1 winner:

- Eliminate 4 people in the first round
- Eliminate 2 people in the first round, and 2 in the second round

In the first case, there are 5 choices for the final winner and 4 choices for who they reveal. Among the four others, there are 3 choices for each reveal. However, we must ensure each person is revealed by at least one other, so

$$3 \cdot 3 \cdot 2^3$$

cases do not work (3 choices for who isn't revealed, 3 choices for who that person reveals and 2 for each of the other 3).

We must add back cases where 2 of the 4 are not revealed; there are

$$\binom{3}{2} \cdot 2^2 \cdot 1^2$$

ways here (3 ways to choose the 2 not revealed, 2 choices for each of them, and 1 for each of the people revealed. With PIE we find:

$$3^4 - 3 \cdot 3 \cdot 2^3 + 3 \cdot 2^2 = 21$$

possible ways in this case. Thus, this probability is $5 \cdot 4 \cdot \frac{21}{4^5} = \frac{105}{256}$.

In the second case, there are $\binom{5}{2} = 10$ ways to choose the first 2 eliminated. Then, there are $2^3 = 8$ ways to assign eliminations. This occurs with probability $10 \cdot \frac{8}{4^5} = \frac{5}{64}$. The second round there are 3 ways to choose the final winner, and 2 ways to assign reveals for a probability of $\frac{3}{4}$. Thus the winning probability here is:

$$\frac{5}{64} \cdot \frac{3}{4} = \frac{15}{256}$$

Overall, the probability of there being 1 winner at the end is $\frac{105}{256} + \frac{15}{256} = \frac{15}{32}$, so $m + n = \boxed{47}$.