Florida Student Association of Mathematics

2022 Spooky Halloween Contest Individual Round Solutions

October-November 2022

1. Wichita arranges her 300 toads along a hallway. The weights of the 300 toads form an arithmetic sequence. The first 100 toads weigh 100 units in total and the combined weight of all the toads is 700 units. What is d^{-1}, where d is the common difference in the arithmetic sequence of weights?

   Answer: 75

   Proposed by Ahan Mishra

   Let the sum of the first n terms be S_n. Then $S_{300} - 3 \cdot S_{100} = (S_{300} - S_{200}) - S_{100} + (S_{200} - S_{100}) - S_{100} + S_{100} - S_{100} = 100 \cdot 200d + 100 \cdot 100d + 100 \cdot 0d$ where the final result is from matching up the corresponding sequence term by term (for example $a_1 + 198d$ versus $a_1 + 98$d). Then we have that $30000d = 400$, so $d = 1/75$, and the answer is $\boxed{75}$.

2. Andrew is giving out treats. For any child, there is a $\frac{2}{3}$ chance he gives out nothing and a $\frac{1}{3}$ chance he gives a treat. A child is tricked when the child before them gets a treat, but they get nothing. What is the expected number of tricked children if Andrew meets 100 children?

   Answer: 22

   Proposed by Andrew Xing

   For every two children, there is a $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ probability that the second one is tricked. Since there are 99 ways to get two children in order $((1,2), (2,3), ...(99,100))$, by linearity of expectation, there we expect $\frac{2}{9} \cdot 99 = \boxed{22}$ children to be tricked.

3. A ghost's favorite three-digit number plus the sum of its digits is a multiple of 99. Find the sum of all possible values of this number.

   Answer: 5004

   Proposed by Jessica Wan

   Suppose the number is ABC, with $A \geq 1$. Then, $99 \mid ABC + A + B + C$. Thus, $9 \mid 2(A + B + C)$, and $A + B + C = 9$, 18, 27. Also, $11 \mid A + C$. Thus, $A + C = 11$.

   So, we must have $B = 7$, and find that $279, 378, 477, 576, 675, 774, 873, 972$ are all solutions. Their sum is $101 \cdot (2 + 3 + \ldots + 9) + 70 \cdot 8 = 4444 + 560 = \boxed{5004}$.

4. Let Y and R respectively be the feet of the altitudes from A and C of acute triangle $\triangle SCA$ such that $AC = 25$, $AY = 24$, and $CR = 20$. Find the area of $\triangle CRY$.

   Answer: 42

   Proposed by Andrew Xing

   Since $\angle CRA = \angle CYA = 90°$, $ARYC$ is a cyclic quadrilateral. $AR = 15$ and $CY = 7$ by Pythagorean Theorem, and now we use Ptolemy's Theorem on $ARYC$. This gives $AC \cdot RY + AR \cdot CY = AY \cdot CR$. Solving for RY gets $RY = 15$. To find the area of CRY, we can use $\frac{1}{2}CY \cdot YR \cdot \sin(\angle CYR)$. Since $ARYC$ is cyclic, we know that $\angle CYR = 180° - \angle CAR$, so

   $$\sin(\angle CYR) = \sin(\angle CAR) = \frac{20}{25} = \frac{4}{5},$$

   and the area of $\triangle CRY$ is $\frac{1}{2} \cdot \frac{4}{5} \cdot 7 \cdot 15 = \boxed{42}$.

5. Ramez and Yuhan go to a fair to celebrate Halloween. There is a restaurant selling pumpkin pies and hot dogs. There are 10 pies and 6 hot dogs, each stored in its own non-distinguishable box. If Ramez

and Yuhan choose 8 boxes randomly, the probability Ramez has more hot dogs than Yuhan can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m+n$.

Answer: 373

Proposed by Tom Zhang

We see that the probability of either Yuhan or Ramez gets more hot dogs is the same. We only need to consider the time that they have same hot dogs and use PIE. The answer is simply

$$\frac{1}{2}\left(1-\frac{\binom{6}{3}\cdot\binom{10}{5}}{\binom{16}{8}}\right)=\frac{87}{286},$$

yielding an answer of $\boxed{373}$.

6. Aaron the sorcerer's favorite function is $f(x)=\frac{1}{3+3^x}$. Given that

$$f\left(\frac{1}{2022}\right)+f\left(\frac{2}{2022}\right)+\cdots+f\left(\frac{4042}{2022}\right)+f\left(\frac{4043}{2022}\right)$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m+n$.

Answer: 4049

Proposed by Andrew Xing

Observe that

$$f(2-x)=\frac{1}{3+3^{2-x}}=\frac{3^{x-1}}{3^x+3}$$

so

$$f(2-x)+f(x)=\frac{3^{x-1}+1}{3^x+3}=\frac{1}{3}$$

Then, pairing outer terms gives 2021 pairs of $\frac{1}{3}$ and $\frac{1}{6}$ from $f(\frac{2022}{2022})$, so summing this up gives $\frac{4043}{6}$, for an answer of $\boxed{4049}$.

7. A quirky pumpkin carver draws a coordinate axis on his pumpkin. He draws the square mouth with a side of the square on the line $y=x+2$ and the opposite side of the square on the curve $x=y^2$. What is the maximum possible area of the mouth square?

Answer: 98

Proposed by Anagh Sangavarapu

Note that the opposite side of the square is determined by the intersection of the curve $x=y^2$ and a line $y=x-k$ for some constant k. The vertices of this side have $y-$coordinates satisfying $y^2=y+k$. From Vieta, the absolute difference between these two roots is $\sqrt{4k+1}$, so the distance between these two points (equivalently, the side length of the square) is equal to $\sqrt{8k+2}$.

Now, the distance between lines $y=x-k$ and $y=x+2$ is $(k+2)/\sqrt{2}$, so setting it equal to $\sqrt{8k+2}$ and solving gives $k=0,12$. The area of the square $8k+2$ is maximized when $k=12$, yielding an answer of $\boxed{98}$.

8. Initially, there is a singular bone in a graveyard. Every minute, the number of bones in the graveyard is multiplied by a random factor of 2022. If the number of bones in the graveyard is a multiple of 2022, all the bones combine to form a skeleton. The expected number of minutes it takes for a skeleton to form can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m+n$.

Answer: 29

Proposed by Aaron Hu

Let E_k denote the expected number of minutes until a skeleton is formed if the number of bones in the graveyard has k distinct prime factors. Then we have

$$\begin{aligned} E_0 &= \frac{1}{8}(E_0 + 3E_1 + 3E_2 + E_3) + 1, \\ E_1 &= \frac{1}{4}(E_1 + 2E_2 + E_3) + 1, \\ E_2 &= \frac{1}{2}(E_2 + E_3) + 1. \end{aligned}$$

Additionally, $E_3 = 0$, so solving gives $E_2 = 2$, $E_1 = \frac{8}{3}$, and $E_0 = \frac{22}{7}$, for a final answer of $\boxed{29}$.

9. Call an ordered pair of positive integers (a, b) *spooky* if it satisfies

$$\gcd(a, b)^2 = \operatorname{lcm}(a, b) + 2023.$$

What is the number of spooky pairs of positive integers?

Answer: 24

Proposed by Aaron Hu

Suppose that $(a, b) = (dx, dy)$, where $d = \gcd(a, b)$. Rearranging the given equation and dividing by d gives

$$\operatorname{lcm}(x, y) = \frac{\operatorname{lcm}(a, b)}{d} = d - \frac{2023}{d} = m,$$

so $d \mid 2023$ and is greater than $\sqrt{2023}$. Since $2023 = 7 \cdot 17^2$, we have

$$(d, m) \in \{(119, 102), (289, 282), (2023, 2022)\}.$$

Note that the number of solutions to $\operatorname{lcm}(x, y) = m$ where x, y are relatively prime is 2^k, where k is the number of distinct prime divisors of m. Then since $102 = 2 \cdot 3 \cdot 17$, $282 = 2 \cdot 3 \cdot 47$, and $2022 = 2 \cdot 3 \cdot 337$, the answer is just $3 \cdot 2^3 = \boxed{24}$.

10. Tiger is performing a magic trick at a Halloween fair! He has 97 numbers $\frac{49}{1}, \frac{49}{2}, \cdots, \frac{49}{97}$ and he asks you to keep replacing two numbers, say a and b, with $2ab - a - b + 1$. Then, he predicts what the last number is! As always, he is right. What was his prediction?

Answer: 1

Proposed by Xuzhou Ren

Observe that $2ab - a - b + 1 = 2(a - \frac{1}{2})(b - \frac{1}{2}) + \frac{1}{2}$. Then, plugging this in again yields $2^2(a - \frac{1}{2})(b - \frac{1}{2})(c - \frac{1}{2}) + \frac{1}{2}$. In general, the $+\frac{1}{2}$ and $-\frac{1}{2}$ cancel, so the final number is

$$2^{96} \prod_{k=1}^{97} \left(\frac{49}{k} - \frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2} + 2^{96} \prod_{k=1}^{97} \frac{98-k}{2k} = \frac{1}{2} + \frac{2^{96} \cdot 97!}{2^{97} \cdot 97!} = \boxed{1}.$$