Florida Student Association of Mathematics

2022 Spooky Halloween Contest Team Round Solutions

October-November 2022

1. Yuhan eats an unusual snack for Halloween that consists of a sliced cucumber with initial mass 300g saturated in Spooky Essence$^{\text{TM}}$, and a sliced cucumber with initial mass 250g saturated in pumpkin spice latte. The cucumbers are 96% water, while the Spooky Essence$^{\text{TM}}$ is 70% water and the pumpkin spice latte is 90% water. Assuming cucumbers are saturated when they absorb 50% of their mass, the fraction of the mass of Yuhan's snack that was water is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Answer: 1047

   Proposed by Yuhan Niu

   The water in the cucumbers is a total of $(300+250)\cdot 0.96 = 528$g. There are 150g of Spooky Essence$^{\text{TM}}$ and 125g of pumpkin spice latte, which contribute 105g and 112.5g of water respectively. The total amount of mass eaten is $(300+250)\cdot(1+0.5) = 825$g. Simplify $\frac{528+105+112.5}{825}$ to $\frac{497}{550}$. The sum of these two is $\boxed{1047}$.

2. Andrew is making Halloween friendship bracelets. For every bracelet, he puts 5 orange beads, 7 black beads, and 11 white beads in a random order on a wire. If the leftmost bead on the wire has place 1, and every place to the right increases by 1, what place is the first orange bead expected to be?

   Answer: 4

   Proposed by Andrew Xing

   Between our 5 orange beads, there are 6 spaces. Since there are 18 other beads to place, each space is expected to have 3 other beads. So, the expected place for the first orange bead is $\boxed{4}$.

3. Giving out candy is very troublesome for Darius. Each time he opens the door to give out candy

   - with a probability of $\frac{4}{5}$, he returns to his chair
   - with probability $\frac{3}{20}$, the doorbell rings on his way back to his chair and he opens the door again
   - and with probability $\frac{1}{20}$, he either gives up giving out candy or runs out and rests for the rest of the night.

   What is the expected number of times he sits back down in his chair before he gives up or runs out and rests for the night?

   Answer: 16

   Proposed by Yuhan Niu

   Let s be the expected number of sits. From that, we can set up the equation $s = \frac{4}{5}(s+1) + \frac{3}{20}\cdot s + \frac{1}{20}0$. Solving gets us $s = \boxed{16}$.

4. To ward away a sphinx, Alex must find the sum of all nonnegative integers n where $9n+16$ and $16n+9$ are both positive squares. What answer should Alex give to survive?

   Answer: 53

   Proposed by Andrew Xing

   We can set $9n+16 = x^2$ and $16n+9 = y^2$. From here, multiplying the first equation by 16 gives $144n+256 = 16x^2$ and multiplying the second equation by 9 gives $144n+81 = 9y^2$. Substituting $144n$ in gets $16x^2 - 175 = 9y^2$. From here, we can factor our difference of squares and look for pairs of x and y that will work. We find that $(22,29)$, $(5,5)$, and $(4,3)$ all work. This gives $n=0, n=1$, and $n=52$ as possible values for n, so $\boxed{53}$ is our final answer.

5. The number $16^{32} + 128$ has a unique 3-digit prime factor. Compute this prime factor.

Answer: 683

Proposed by Karthik Vedula

Note that

$$16^{32} + 128 = 2^{128} + 128 = 128(2^{121} + 1) = 128(2^{11} + 1)(2^{110} - 2^{99} + 2^{88} - \cdots + 2^{22} - 2^{11} + 1)$$

We have $2^{11} + 1 = 2049 = 3 \cdot 683$, and so $\boxed{683}$ must be our answer.

6. Call a pair of distinct, positive real numbers (a, b) *compatible* if the following statement is true:

$$a + \cfrac{b}{a + \frac{b}{a+\dots}} = b + \cfrac{a}{b + \frac{a}{b+\dots}}.$$

Over compatible (a, b), the minimum of $a^2 + 4b^2$ is $\frac{m}{n}$ for coprime, positive integers m, n. Compute $m^2 + n^2$.

Answer: 41

Proposed by Karthik Vedula

We first classify all compatible pairs. Let S denote the common value of the continued fractions. We have

$$a + \frac{b}{S} = b + \frac{a}{S} \implies \frac{b-a}{S} = b - a \implies S = 1.$$

This means that $a + \frac{b}{S} = S$ turns into $a + b = 1$, which is both necessary and sufficient (including $a \neq b$). We now have

$$a^2 + 4b^2 = (1-b)^2 + 4b^2 = 5b^2 - 2b + 1 = 5\left(b - \frac{1}{5}\right)^2 + \frac{4}{5} \geq \frac{4}{5}.$$

Equality can occur when $(a, b) = (0.8, 0.2)$. Our answer is thus $4^2 + 5^2 = \boxed{41}$.

7. Algebra problems are scary right? When solved over reals,

$$\left(x^2 + 1\right)\left(y^2 + 6\right)\left(z^2 + 24\right) = 96xyz$$

has solutions at $(x_1, y_1, z_1), (x_2, y_2, z_2) \dots (x_n, y_n, z_n)$. Compute $\prod_{i=1}^{n} x_i y_i z_i \pmod{1000}$.

Answer: 736

Proposed by Xuzhou Ren

By AM-GM on each term we see that this is just the AM-GM equality case as long as $xyz \geq 0$. Therefore, $x^2 = 1, y^2 = 6, z^2 = 24$. There are 2 possible cases, namely $x, y, z \geq 0$ or two of them are negative and one is positive. Each case has the same product and the second case contributes 3 triples, so $12^4 \equiv \boxed{736} \pmod{1000}$.

8. A ghost goes trick-or-treating in an infinite neighborhood numbered $3, 6, 9, \dots$ Each house it knocks on will only give it candy if the number of previous houses who have treated the ghoul (with candy!) divides its house number, though the first house (number 3) offers it candy regardless. Find the number of pieces of candy it will have collected after visiting 2022 houses.

Answer: 1012

Proposed by Jessica Wan

We examine the pattern from the first few houses. Let a_{3n} be the number of pieces of candy the ghoul has collected after knocking on house 3n. A house offers candy iff $a_{3n-3} \mid 3n$.

House i	3	6	9	12	15	18	21	24	27	30	33	36	39
Treat	✓	✓	X	✓	✓	X	X	✓	X	✓	X	✓	X
a_i	1	2	2	3	4	4	4	5	5	6	6	7	7

Note that a pattern forms by a_{24}, and one can prove by induction that after house 24, only houses with numbers divisible by 6 will give the ghost candy. This is because, by induction, $a_{6i} = a_{6i+3} = i + 1$ for $i \geq 4$. Thus, as $i + 1$ doesn't divide $6i + 3$ for $i \geq 4$ and it clearly divides $6i + 6$, our claim is proven.

As 4 houses up to number 21 give it candy, the ghost will collect $\boxed{1012}$ candies, from houses {3, 6, 12, 15, 24, 30, ..., 6066}.

9. Quadrilateral $GHST$ is inscribed in a circle with center O and radius 22. Given that $\overline{GT} \parallel \overline{HS}$, $\angle HOS = 90^\circ$, and pentagon $GHOST$ is convex, what is the difference between the maximum and minimum possible areas of $GHOST$?

Answer: 121

Proposed by Aaron Hu

Let $\alpha = \angle HOG$, $\beta = \angle GOT$, and $\gamma = \angle TOS$. Then

$$[GHOST] = \frac{22^2}{2}(\sin\alpha + \sin\beta + \sin\gamma),$$

where $\alpha + \beta + \gamma = \pi/2$. Note that $[GHOST] \geq [HOS]$, with equality occurring when $GHST$ is degenerate. This yields a minimum of $22^2/2 = 242$. Intuitively, the area is maximized when $\alpha = \beta = \gamma = \pi/6$, which follows from, say, Jensen's inequality. This yields a maximum area of $(22^2/2)(3/2) = 363$, and an answer of $363 - 242 = \boxed{121}$.

10. In concave equilateral hexagon $HAUNTS$, we have $\angle A = 108^\circ$ and $\angle H = \angle U = 96^\circ$. Find $\angle T$ given that it is less than 180°.

Answer: 36

Proposed by Aaron Hu

Let X, Y, and T' be points outside the hexagon such that $HAUXY$ is a regular pentagon and $\triangle XYT'$ is equilateral. Then

$$\angle SHY = \angle AHY - \angle AHS = 12^\circ, \quad \angle HYT' = \angle HYX + \angle XYT' = 168^\circ,$$

so $\overline{SH} \parallel \overline{TY'}$. Since $T'Y = YH = HS$, we have that $T'YHS$ is a rhombus. Similarly, $T'XUN$ is a rhombus, so

$$T'S = SH = NU = T'N,$$

implying that $T = T'$. Now,

$$\angle T = \angle XTY - \angle XTN - \angle YTS = 60^\circ - 2 \cdot 12^\circ = \boxed{36^\circ},$$

as desired.