Florida Student Association of Mathematics

2022-2023 Everything Tryout

January 21-22, 2023

Round 3: Combinatorics

1. Alex flips 12 fair coins and rolls a fair 12-sided die. The probability that the number he rolls on the die equals the number of coins that turn up heads can be expressed as $\frac{m}{2^n}$ for positive integers m, n with m odd. Compute $m + n$.

2. A bug is on a triangle with vertices labeled 1, 7, and 17, starting at the vertex labeled 1. Every minute, the bug moves to a random adjacent vertex. The bug does this until the product of the labels of the vertices it has visited is at least 2023. Given that the probability that this product is exactly 2023 when the bug stops can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m + n$.

3. Let N be the number of ways there are to place 0's and 1's in a 5×5 grid of squares such that the sum of numbers in every domino (over all dominoes) is a multiple of 4. Compute the number of positive integer factors of N.

4. Compute the number of permutations of the list $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that one can split the permutation into two contiguous, non-empty lists such that the product of both lists is relatively prime. (For example, one such permutation is $\{2, 3, 4, 1, 6, 8, 5, 7\}$, because it can be split into $\{2, 3, 4, 1, 6, 8\}$ and $\{5, 7\}$.)

5. Call a function $f\colon \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 16\}$ *beautiful* if it satisfies the following: if $f(x) = f(y)$, then $x = y$. Compute the least positive integer N such that the following is true for all beautiful functions: for integers $1 \le m, n \le 16$, if there exists a positive integer ℓ such that $f^{\ell}(m) = n$, then $f^N(m) = n$. (Here, $f^1(x) = f(x)$, and $f^a(x) = f^{a-1}(f(x))$ for integers $a \ge 2$.)

6. Karthik rolls 80 fair four-sided die (each with faces labeled 1 through 4) and multiplies all the numbers that turn up together. What is the expected number of factors his product has?