Florida Student Association of Mathematics

2022-2023 Everything Tryout

January 21-22, 2023

Solutions

Algebra and Number Theory

1. The number 3375343 is the product of two primes. Compute the smaller one.

   Answer: 157

   Proposed by Karthik Vedula

   Recognize that $3375 = 15^3$ or bash primes until your next birthday, and we see that $3375343 = 150^3 + 7^3$, so $150 + 7 = \boxed{157}$ is the prime factor we are looking for.

2. Let $a > 1$ be a real number such that $r = a^3 + \frac{1}{a^3}$ and $s = a^2 + \frac{1}{a^2}$ are both integers. Compute the minimum value of rs.

   Answer: 126

   Proposed by Karthik Vedula

   Note that $a^6 + \frac{1}{a^6}$ is equal to $r^2 - 2$ and $s^3 - 3s$ by expansion. This means $r^2 = s^3 - 3s + 2 = (s-1)(s^2 + s - 2) = (s-1)^2(s+2)$. This means $s + 2$ is a perfect square, and $s > 2$. This means $s = 7$, and thus $r = 18$, giving $rs = \boxed{126}$.

3. Let 2^N be the largest power of 2 which divides $1! \cdot 2! \cdot 3! \cdots 63!$. Compute N.

   Answer: 1824

   Proposed by Karthik Vedula

   Consider the alternative version to Legendre's formula: $v_2(n!) = n - s_2(n)$, where s_2 denotes the sum of the digits in base 2. Doing this for all of $n = 1, 2, \dots, 63$ gives

   $$v_2(1!2!3!\dots 63!) = 1 + 2 + \cdots + 63 - (s_2(0) + s_2(1) + s_2(2) + \cdots + s_2(63)) =$$

   $$\frac{1}{2} \cdot 63 \cdot 64 - \frac{1}{2} \cdot \log_2(64) \cdot 64 = 32(63 - 6) = 32(57) = \boxed{1824}.$$

4. Let F_k denote the kth Fibonacci number defined by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k > 2$, and let P be the unique polynomial with degree 99 satisfying $P(k) = F_k$ for $k = 1, 2, \dots, 100$. Given that $P(101)$ can be expressed as $F_m + n$ for nonnegative integers m, n with $n < F_{m-1}$, compute $m^2 + n$.

   Answer: 10000

   Proposed by Aaron Hu

   From finite differences, we get that the first set of differences is $F_0, F_1, \dots, F_{98}$, the second set is $F_{-1}, F_0, \dots, F_{96}$, and so on until the 99th set of differences F_{-98}. From here, it is clear that

   $$P(101) = F_{-98} + F_{-96} + F_{-94} + \cdots + F_{98} + F_{100} = F_{100},$$

   as $F_{-k} = -F_k$ for even k. Then the answer is $\boxed{10000}$.

5. Consider function f satisfying $f(0) = 0$ and $f(n) = f(n-p) + 1$ for all integers $n > 1$, where p is the smallest prime divisor of n. Find the sum of all k such that the number of composite solutions to $f(n) = k$, $1 < n \leq 500$ is maximized.

   Answer: 911

   Proposed by Aaron Hu

Note that n and p have the same parity, so $n - p$ is always even. Then

$$f(n) = f(n-p) + 1 = \frac{n-p}{2} + 1.$$

If $f(n) = k$ has solutions $n_1, \dots, n_M$ and smallest prime factors $p_1, \dots, p_M$, respectively, then

$$d = n_1 - p_1 = n_2 - p_2 = \cdots = n_M - p_M$$

for some positive integer d. Since each p_i divides n_i, d is divisible by all p_i. Clearly, the p_i are distinct, so

$$p_1 p_2 \dots p_M \mid d \implies d \geq p_1 p_2 \dots p_M.$$

Since $n_i > d$, the largest value of M is 4, as

$$2 \cdot 3 \cdot 5 \cdot 7 < 500 < 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11.$$

The solutions to $f(n) = k$ are of the form $a \cdot P + p_i$ for $1 \leq i \leq 4$ and some positive integer a, where $P = p_1 p_2 p_3 p_4$. Since $P < 500$, we can determine that only $P = 210, 330, 390, 462$ give valid k. For $a = 1$, we get $k = \frac{a \cdot P}{2} + 1 = 106, 166, 196, 232$, respectively. Only $P = 210$ gives a solution for $a = 2$, yielding $k = 211$, so the answer is $\boxed{911}$, as desired.

6. Let $P(x) = 2(x-1)^2$. Given that there are n values of $x \geq \frac{1}{2}$ satisfying

$$\underbrace{P(P(\dots P}_{2023\ P'\text{s}}(x)\dots)) = x,$$

find the sum of the digits in the base-2 representation of n.

Answer: 1012

Proposed by Aaron Hu

The key observation is that

$$P(\cos\theta + 1) = 2\cos^2\theta = \cos 2\theta + 1,$$

so the equation becomes $\cos 2^{2023}\theta = \cos\theta$. Then the set of all possible values of $\cos\theta$ is

$$\left\{\cos\frac{2k\pi}{2^{2023}-1} \;\middle|\; 0 \leq k < 2^{2023} - 1\right\} \bigcup \left\{\cos\frac{2k\pi}{2^{2023}+1} \;\middle|\; 0 \leq k < 2^{2023} + 1\right\},$$

which is just

$$\left\{\cos 0, \cos\frac{2\pi}{2^{2023}-1}, \dots, \cos\frac{(2^{2023}-2)\pi}{2^{2023}-1}, \cos\frac{2\pi}{2^{2023}+1}, \dots, \cos\frac{2^{2023}\pi}{2^{2023}+1}\right\}.$$

Note that $x \geq \frac{1}{2}$ for $\cos\theta \geq \cos\frac{2\pi}{3}$, so

$$n = 1 + \frac{2^{2023}-2}{3} + \frac{2^{2023}+1}{3} = \frac{2^{2024}-1}{2^2-1} + 1,$$

which equals

$$2^{2022} + 2^{2020} + \cdots + 2^2 + 2^0 + 1 = 2^{2022} + 2^{2020} + \cdots + 2^2 + 2^1,$$

so the answer is $2022/2 + 1 = \boxed{1012}$.

Geometry

1. Let $ABCD$ be a rhombus with side length 6. Given that the angle bisectors of $\angle ABD$ and $\angle ACD$ intersect on $\overline{AD}$, compute $\mathcal{A}^2$, where $\mathcal{A}$ is the area of $ABCD$.

   Answer: 324

   Proposed by Aaron Hu

   From Angle Bisector Theorem, we have $\frac{AB}{BD} = \frac{AC}{CD}$, so

   $$[ABCD] = \frac{1}{2} \cdot AC \cdot BD = \frac{1}{2} \cdot AB \cdot CD = \frac{1}{2}(6)(6) = 18,$$

   so the answer is $18^2 = \boxed{324}$.

2. Consider $\triangle ABC$ with $AB = 4$ and $AC = 8$. Given that the length of the angle bisector of $\angle BAC$ contained inside the circumcircle of $\triangle ABC$ is 12, find BC^2.

   Answer: 112

   Proposed by Aaron Hu

   Let the angle bisector of $\angle BAC$ intersect $\odot(ABC)$ at D. Then $DB = DC = x$, so from Ptolemy,

   $$12x = 4DC + 8DB = AD \cdot BC = 12BC,$$

   implying that $\triangle DBC$ is equilateral. Then $\angle BAC = 120^\circ$, so from Law of Cosines, $BC^2 = 4^2 + 8^2 + 4 \cdot 8 = \boxed{112}$.

3. Concave pentagon $ABCDE$ has side lengths all equal to 1 and $\angle D > 180^\circ$. Additionally, $AC = BE = \sqrt{1+\sqrt{3}}$. Compute $\angle D$, in degrees.

   Answer: 240

   Proposed by Karthik Vedula

   Note that $ABCE$ is an isosceles trapezoid, so $AC \cdot BE = AE \cdot BC + AB \cdot CE \implies 1 + \sqrt{3} = 1 + CE$, so $CE = \sqrt{3}$. This means the non-reflex $\angle CDE$ is 120° (1-1-$\sqrt{3}$ triangle), so our answer is $\boxed{240}$.

4. The parabolas $6y = x^2 - 100$ and $8x = y^2 - 100$ intersect in quadrants 2 and 4 of the coordinate plane at A and B, respectively. The perpendicular bisector of $\overline{AB}$ passes through a unique lattice point (a, b). Compute $a^3 + b^3$.

   Answer: 91

   Proposed by Karthik Vedula

   Note that adding the equations gives that the four intersection points of the two parabolas lie on the circle $8x + 6y = x^2 + y^2 - 200$, which has center $(4, 3)$. The perpendicular bisector of AB must pass through this point, so our answer is $4^3 + 3^3 = \boxed{91}$.

5. In unit square $ABCD$, equilateral triangles ABE and ABF are erected outside and inside the rectangle, respectively. Line CF intersects the circumcircle of DEF again at $S \neq F$, and line DF intersects the circumcircle of CEF again at $T \neq F$. Find the integer closest to ST^3.

   Answer: 20

   Proposed by Karthik Vedula

Note that $AE = AD$, so $\angle AED = 15°$, and $\angle CED = 30°$. Additionally, $AF = AD$, so $\angle AFD = 75°$, and $\angle CFD = 150°$. Thus F is the orthocenter of CED. Now, S is the reflection of C over ED, and T is the reflection of D over EC. This means that SCD and TCD both have angles $15°, 15°, 150°$, so ASD and BTC are equilateral. Thus, $ST = 1 + 2 \cdot \frac{\sqrt{3}}{2} = 1 + \sqrt{3}$, and $ST^3 = (1 + \sqrt{3})^3 = 10 + \sqrt{108}$, which is closest to $\boxed{20}$.

6. Triangle $\triangle ABC$ has side lengths $AB = 13$, $BC = 14$, and $AC = 15$, as well as circumcircle ω. The sides of $\triangle ABC$ partition ω into 4 regions: 1 triangle, and 3 circular segments bounded by a side opposite from a vertex, and a minor arc. Call these regions R_A, R_B, and R_C based on which vertex the side is opposite of (e.g. R_A is bounded by side $\overline{BC}$). Circles ω_A, ω_B, and ω_C are the circles inscribed in R_A, R_B, and R_C, respectively, with maximal area.

   The external tangents of ω_B and ω_A intersect at B_1, and the external tangents of ω_C and ω_A intersect at C_1. The circle centered at B_1 passing through C and the circle centered at C_1 passing through B intersect inside $\triangle ABC$ at A_1. Compute AA_1^2.

   Answer: 65

   Proposed by Karthik Vedula

   Let X, Y, Z denote the midpoints of minor arcs $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$, respectively. From Monge on ω_A, ω_B, and ω, we have that B_1 lies on XZ. From Fact 5, $XI = XC$ and $ZI = ZC$, so XZ is the perpendicular bisector of $\overline{CI}$, where I is the incenter of $\triangle ABC$. Similarly, XY is the perpendicular bisector of $\overline{BI}$, implying that $A_1 = I$. From here, we simply compute

   $$AA_1^2 = (s - a)^2 + r^2 = 7^2 + 4^2 = \boxed{65},$$

   as desired.

Combinatorics

1. Alex flips 12 fair coins and rolls a fair 12-sided die. The probability that the number he rolls on the die equals the number of coins that turn up heads can be expressed as $\frac{m}{2^n}$ for positive integers m, n with m odd. Compute $m + n$.

   Answer: 1379

   Proposed by Aaron Hu

   As long as Alex doesn't get 0 heads, he has a $\frac{1}{12}$ chance of rolling the number of heads on the die. Then the desired probability is

   $$\frac{1}{12}\left(1 - \frac{1}{2^{12}}\right) = \frac{1365}{2^{14}},$$

   so the answer is $1365 + 14 = \boxed{1379}$.

2. A bug is on a triangle with vertices labeled 1, 7, and 17, starting at the vertex labeled 1. Every minute, the bug moves to a random adjacent vertex. The bug does this until the product of the labels of the vertices it has visited is at least 2023. Given that the probability that this product is exactly 2023 when the bug stops can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, compute $m + n$.

   Answer: 47

   Proposed by Aaron Hu

   We will count all cases where the product equals 2023 and add the probabilities of getting each case. Note that the bug must visit 7 once and 17 twice. Then the bug can visit 1 anywhere from 1 to 3 times, as after it visits 1, it must go to one of 7 or 17. We have the following cases:

- If it visits 1 once, the path it takes must be $1-17-7-17$, with probability $\frac{1}{8}$.
- If it visits 1 twice, the path it takes must be one of $1-17-1-7-17$, $1-17-1-17-7$, $1-7-17-1-17$, or $1-17-7-1-17$, each with probability $\frac{1}{16}$, for a total probability of $\frac{1}{4}$ for this case.
- If it visits 1 three times, the path it takes must be of the form $1-x-1-y-1-z$, where x, y, z is some permutation of $7, 17, 17$. Each path has a probability of $\frac{1}{32}$ of being taken, so the probability for this case is $\frac{3}{32}$.

Then the desired probability is $\frac{1}{8}+\frac{1}{4}+\frac{3}{32}=\frac{15}{32}$, for an answer of $\boxed{47}$.

3. Let N be the number of ways there are to place 0's and 1's in a 5×5 grid of squares such that the sum of numbers in every domino (over all dominoes) is a multiple of 4. Compute the number of positive integer factors of N.

Answer: 24

Proposed by Karthik Vedula

The number in a corner contributes 2 times its contents in this sum, and the number in an edge square adjacent to this corner contributes 3 times its contents in this sum. Therefore, if we select all but these two numbers arbitrarily, then there is exactly one choice out of $\{2(0)+3(0), 2(0)+3(1), 2(1)+3(0), 2(1)+3(1)\} \equiv \{0,1,2,3\} \pmod 4$ which works. Thus, $N = 2^{5^2-2} = 2^{23}$, and our answer is $\boxed{24}$.

4. Compute the number of permutations of the list $\{1,2,3,4,5,6,7,8\}$ such that one can split the permutation into two contiguous, non-empty lists such that the product of both lists is relatively prime. (For example, one such permutation is $\{2,3,4,1,6,8,5,7\}$, because it can be split into $\{2,3,4,1,6,8\}$ and $\{5,7\}$.)

Answer: 25920

Proposed by Karthik Vedula

Note that $2,4,6,8$ must be on the same side, and $3,6$ must be on the same side of the partition. This means that if both the ends of the permutation contain these numbers, it cannot work. Therefore, at least one of the edges is 1, 5, or 7. All of these permutations clearly work (by chopping off the corresponding end).

We now compute the complement. If both ends are in $\{2,3,4,6,8\}$, there are $5 \cdot 4 \cdot 6!$ ways to choose the ends and center, respectively. Thus, our answer is $8! - 5 \cdot 4 \cdot 6! = 6!(8 \cdot 7 - 5 \cdot 4) = 720(36) = \boxed{25920}$.

5. Call a function $f\colon \{1,2,\ldots,16\} \to \{1,2,\ldots,16\}$ *beautiful* if it satisfies the following: if $f(x) = f(y)$, then $x = y$. Compute the least positive integer N such that the following is true for all beautiful functions: for integers $1 \le m, n \le 16$, if there exists a positive integer ℓ such that $f^{\ell}(m) = n$, then $f^N(m) = n$.

Answer: 720720

Proposed by Karthik Vedula

Consider the directed graph generated with nodes as the domain of f and edges as $x \to f(x)$ for all x in the domain. By the definition of a beautiful function, no two nodes point at each other, so everything is pointed at exactly once. This means that this graph is a collection of disjoint cycles (for any beautiful function).

Now, our goal is to find the least possible period for all possible cycles in a beautiful function. We can have a cycles of length 16, connecting all the numbers, all the way to a cycle of length 1. Thus, our answer is $\operatorname{lcm}(1,2,3,\ldots,16) = \boxed{720720}$.

6. Karthik rolls 80 fair four-sided die (each with faces labeled 1 through 4) and multiplies all the numbers that turn up together. What is the expected number of factors his product has?

Answer: 1266

Proposed by Aaron Hu

From linearity of expectation, we want to find the value of

$$\mathbb{E}[(a+1)(b+1)] = \mathbb{E}[ab] + \mathbb{E}[a] + \mathbb{E}[b] + 1,$$

where the final product is of the form $2^a \cdot 3^b$. The expected contribution of each die to a, b is $\frac{3}{4}, \frac{1}{4}$, respectively, so

$$\mathbb{E}[a] + \mathbb{E}[b] = 60 + 20 = 80.$$

It suffices to compute $\mathbb{E}[ab]$. Let the die turn up numbers $a_1, a_2, \ldots, a_{80}$, and suppose that there are x, y, and z amounts of die that turn up $2, 3, 4$, respectively. Then

$$ab = y(x+2z) = (\underbrace{1+\cdots+1}_{x\text{ 1's}} + \underbrace{2+\cdots+2}_{z\text{ 2's}})(\underbrace{1+\cdots+1}_{y\text{ 1's}}) = \sum_{1\le i<j\le 80} f(a_i a_j),$$

where $f(k)$ equals the number of divisors of k that are multiples of six. Then from linearity of expectation,

$$\mathbb{E}[ab] = \sum_{1\le i<j\le 80} \mathbb{E}[f(a_i a_j)] = \frac{6}{16}\binom{80}{2} = 1185,$$

so the answer is $1185 + 80 + 1 = \boxed{1266}$.