2016 FLSAM PUMaC Tryout

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1. Do not look at the test before the round begins.
2. This test consists of 16 short answer problems to be solved in 2 hours. Problems are given in four sets of four, where each set consists of a problem in Algebra, Geometry, Combinatorics, and Number Theory. The problems within each set are of approximately equal difficulty, the sets are ordered in increasing difficulty. There is no penalty for incorrect answers.
3. The problems are weighted by set. Set IV is worth more than Set III, which is worth more than Set II, and so on.
4. Write your name, answers, and other requisite information on the accompanying answer sheet. Also include, if applicable and to the best of your memory, your scores on the 2016 AMC series of tests, making sure to circle whether you took the AMC 10 or AMC 12, and the AIME I or AIME II.
5. No computational aids other than pencil/pen are permitted.
6. Answers are not necessarily integers. All fractional answers should be reduced, and radicals/logarithms should be expressed in simplest possible form.

Answer Sheet

Name: ________________________________

School: ________________________________

Grade: ________________________________

Gender: **M** **F**

2016 AMC (**10** / **12**) A: ____________________

2016 AMC (**10** / **12**) B: ____________________

2016 AIME (**I** / **II**): ______________________

Problem	Answer
A1	
G1	
C1	
N1	

Problem	Answer
A2	
G2	
C2	
N2	

Problem	Answer
A3	
G3	
C3	
N3	

Problem	Answer
A4	
G4	
C4	
N4	

Set I

A1. If r, s, and t are the roots of the cubic equation $x^3 - 6x^2 + 6x + 2 = 0$, find the value of $r^3 + s^3 + t^3$.

G1. My dog loves to chase squirrels around the back yard, bounded by fences on the line $x = 25$, $x = 0$, and $y = 20$, while my house's wall lies on $y = 0$. One day, my dog raced a squirrel from my back door, located at $(8, 0)$, to touch each fence once before the finish line at a tree at $(13, 12)$ (when the squirrel escapes up the tree). What is the shortest possible distance my dog could have run?

C1. A room contains 1 first grader, 4 second graders, 9 third graders, 16 fourth graders, and 25 fifth graders. What is the minimum number of students that the principal must pick in order to guarantee that among those picked are 10 students who are all from the same grade?

N1. How many positive integral values of $k \leq 100$ produce a positive integral solution (x, y) to the equation $x^3 - y^3 = xy + k$?

Set II

A2. Define a sequence $\{a_n\}$ such that $a_1 = 20$, $a_2 = 16$, and for $n \geq 3$, $a_n = \frac{2}{3}a_{n-1} + \frac{1}{4}a_{n-2}$. Evaluate $\sum_{n=1}^{\infty} a_n$.

G2. Aaron the Ant is walking on regular tetrahedron $ABCD$, which has side length 3. Let X be the point on side BC such that $BX = 1$, let Y be the point on side AB such that $AY = 1$, and let Z be the intersection of AX and CY. If Aaron starts at point Z and wants to walk to point D, only walking on the surface of the tetrahedron, find the shortest possible distance that he must walk.

C2. Ben wants to pick a random decimal digit, so he comes up with the following construction. Firstly he rolls a fair 10-sided die with sides labeled 1 through 10, and records the result k. Then he rolls a fair k-sided die with sides labeled 0 through $k - 1$, and the result is his final digit. If p is the probability that this digit is even, find $\lfloor 1000p \rfloor$.

N2. Let $S_b(n)$ denote the sum of the digits of the base b representation of some base 10 integer n. For example, $S_2(7) = 3$ and $S_4(7) = 4$. Let K be the smallest positive integer satisfying $\frac{S_2(K)}{S_4(k)} = \frac{2017}{3034}$. Evaluate $S_8(K)$.

Set III

A3. Let $\{a_i\}$ be a sequence of positive real numbers satisfying $a_1 = 9$ and $a_{n+1}^{a_n - \ln(a_n)} = a_n^{a_n}$ for all positive integers n. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{a_n}.$$

G3. In trapezoid $ABCD$, $AB||CD$, $AB = 20, BC = 4, CD = 6, DA = 16$. A circle with center P on segment AB is tangent to both AD and BC. Find the length of AP.

C3. Albert, Blaise, and Carl like to go to the gym every morning. Each morning, Albert chooses a time uniformly at random to arrive at the gym between $6:00$ and $7:00$. Blaise chooses a time uniformly at random to arrive at the gym between $6:30$ and $7:30$. Carl chooses a time uniformly at random to arrive at the gym between $7:00$ and $8:00$. Given that they always remain at the gym for exactly one hour in the morning, what is the probability that all three are in the gym simultaneously at some time on any given morning?

N3. Let u and v be two-dimensional vectors with integer coordinates. Given that $u \cdot v = 2016$, find the number of possible integer values of $|u| \cdot |v|$.
(For $u = (a, b)$ and $v = (c, d)$, we define $u \cdot v = ac + bd$ and $|u| = \sqrt{a^2 + b^2}$.)

Set IV

A4. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a surjective, non-decreasing function satisfying

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{9}{7}.$$

Find $f(2016)$. [Hint: The set $\mathbb{Z}^+$ is the set of positive integers. The function f is surjective if and only if for every $n \in \mathbb{Z}^+$, there is some $m \in \mathbb{Z}^+$ such that $f(m) = n$.]

G4. Let S be the set of points (x, y) in the Cartesian plane for which $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is a lattice point. Find the largest possible area of a triangle with vertices in S and orthocenter at the origin.

C4. A row of holes starts with hole 1 and continues infinitely to the right, labeled with the positive integers. Some finite number of these holes have pegs in them. A series of operations is performed as follows: if holes n and $n + 1$ have pegs and hole $n + 2$ is empty, then we can move the peg in hole n to hole $n+2$ and remove the peg in hole $n+1$. The *result* of an initial distribution of pegs is the distribution reached when as many operations as possible have been performed. Two initial distributions are considered equivalent of they have the same results. Find the size of the largest possible set of pairwise non-equivalent initial distributions with at least one peg and no pegs past hole 10.

N4. Define a positive integer n to be *jovial* if there exist distinct positive integers a, b, c, and d such that a divides b, b divides c, c divides d, and $a + b + c + d = n$. Find the sum of the three largest positive integers that are not jovial.