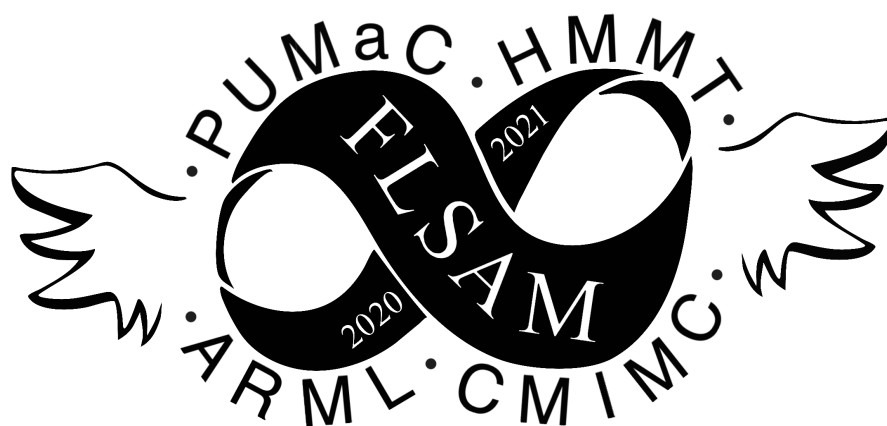


Florida Student Association of Mathematics



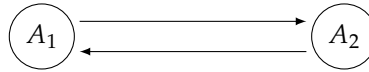
2020 Spooky Halloween Contest Boss Fight

October 2020

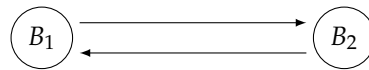
As you enter the password to the rusty gates of the graveyard, the Supreme Spooky Skeleton confronts you! To defeat it, you must work with your team to accumulate as many points as possible; each point deals one point of damage to the Skeleton, who needs $4 \times$ [the number of people at your meeting] points to be defeated, with all teams' scores accumulated. Note that you are only permitted to communicate with your team.

This year, the Skeleton has the extra-spooky ability to warp time! The test will consist of four time loops, A, B, C , and D . The number of problems in each loop are 2, 2, 3, 5, respectively, and every problem will depend on the answer to some other problem in its time loop. The answer to problem n in time loop X will be denoted X_n ; for example, the answer to the first problem of loop A is written as A_1 . Each problem is worth three points, and completing all problems in a time loop correctly will give you bonus points, equal to the number of problems in the loop.

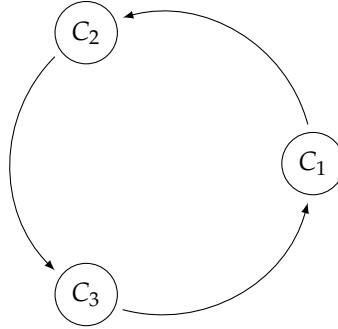
You will have 30 minutes to complete the test. Good luck, and have fun!

Time Loop A [2 Problems]

- A1.** Right triangle $\triangle PHI$ has unit square $TIME$ inscribed into it with TI on HI , IM on PI , and E on PH . The altitude from I to PH is A_2 . Find one-fourth the length of PH .
- A2.** A circle is inscribed in equilateral triangle $\triangle XYZ$, and its radius is A_1 . Find half the side length of $\triangle XYZ$.
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Time Loop B [2 Problems]

- B1.** You start with 0 pumpkins. Each turn, you may add 1 or 10 pumpkins, or multiply your number of pumpkins by 10. Your goal is to make B_2 pumpkins. Given that the minimum possible number of turns is m and the maximum possible number of turns is M , find $M + m$.
- B2.** If B_1 is a four digit integer written as \overline{abcd} and digit sum greater than 10, then find the four digit integer \overline{dabc} .
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Time Loop C [3 Problems]

C1. The sum

$$1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (\mathbf{C}_3 - 1)^2 \cdot \mathbf{C}_3!$$

can be expressed as $a \cdot b! + c$ for positive integers a, b, c where $b! > c$, and $b \geq a$. Find $a + b + c$.

C2. Given that $\sqrt[3]{a} + \sqrt[3]{b} = \sqrt[3]{\mathbf{C}_1}$ for nonzero reals a, b , find the maximum value (an integer) of

$$\frac{\mathbf{C}_1 - a - b}{\sqrt[3]{ab}}.$$

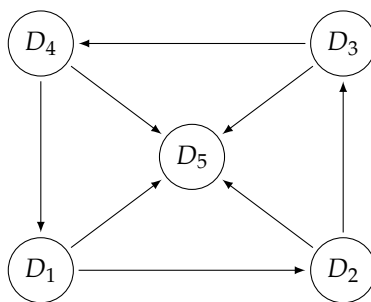
C3. The Supreme Spooky Skeleton has an infinite pond, with lilypads numbered $0, 1, 2, 3, \dots$. He has two valid moves:

- If the largest number lilypad occupied by a frog is K , then for 3 spooky dollars, he can place a frog on an empty lilypad with number less than K .
- If lilypads $k - 2, k - 1$ have frogs and lilypad k does not have a frog, then for 5 spooky dollars he can remove all frogs on lilypads $k - 2, k - 1$ and place a frog at $k + 1$.

His pond starts with two frogs, with one on lilypad 0 and the other on lilypad 1. He wants each of lilypads

$$1, 3, 5, \dots, \mathbf{C}_2 - 2^2$$

to have a frog. What is the minimal number of spooky dollars that the Supreme Spooky Skeleton needs to spend to achieve this?

Time Loop D [5 Problems]

- D1.** Let d be the last digit of \mathbf{D}_4 . Sam the skeleton is the annual Halloween DJ for the Raveyard, and needs to play d songs in sequence, one at a time. However, there are distinct 3 special requests, each of the form (i, j) requiring song i to be played before song j . Across all sets of 3 special requests that can happen such that Sam is able to satisfy all of them, there could be k ways to play all the songs. What is the sum of all possible distinct values of k ?
- D2.** Steve is building an extremely economical Minecraft graveyard, with each of the \mathbf{D}_1 point-sized tombstones placed at a lattice point (x, y, z) , and for some integers p, q, r every lattice point (x, y, z) , $1 \leq x \leq p, 1 \leq y \leq q, 1 \leq z \leq r$, there is a tombstone. For a choice of (p, q, r) , the smallest rectangular prism by volume containing all tombstones on its boundary or interior has volume K ; maximize K .
- D3.** Consider the last four digits of \mathbf{D}_2 to be \overline{pqrs} . Let $M = \overline{pq}$ and $N = \overline{r0}$. a, b , and c are three positive integers such that $a + b + c = N$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{25M}{abc} = 1$. What is the value of $abc - 2$?
- D4.** Your team is now attempting to communicate with the ghosts through the use of a Luigi board, which outputs a single character from a set of \mathbf{D}_3 characters, chosen randomly. This set contains the characters A, B, C . After repeatedly outputting letters and writing them down in sequence, the probability that the Luigi board spells out "AAA" before "ABC" can be represented as a simplified fraction $\frac{m}{n}$. Find n .
- D5.** Take each of your four answers $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4 \bmod 10^4$, and add leading zeroes until they are four digits long. If the leftmost j th digit of \mathbf{D}_i is nonzero, then cross out the corresponding letter in the grid below in row i , column j . Read the remaining letters to discover the Supreme Spooky Skeleton's key weakness!

S	N	C	A
U	M	A	K
L	E	S	C
I	U	M	T